March 9 Math 3260 sec. 55 Spring 2020

Section 4.3: Linearly Independent Sets and Bases

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H =$Span$(\mathcal{B})$.

We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in H is contained in the basis, and none of this information is repeated.

Standard Basis in $\mathbb{R}^n$

The columns of the $n \times n$ identity matrix provide an obvious basis for $\mathbb{R}^n$. This is called the **standard basis** for $\mathbb{R}^n$. For example, the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$ are

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{respectively.}$$

Some Other Vector Spaces

- $\{1, t, t^2, t^3\}$ is the *standard basis* for $\mathbb{P}_3$
- The set $\{1, t, \ldots, t^n\}$ is called the *standard basis* for $\mathbb{P}_n$.
- The set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M^{2\times 2}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Example

Define a *standard basis* for $M^{2\times 3}$.

A typical vector looks like $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

Take the set

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Prelude to a Spanning Set Theorem

Example: Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be vectors in a vector space V, and suppose that

(1) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and
(2) $\mathbf{v}_3 = \mathbf{v}_1 - 2\mathbf{v}_2$.

Show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Let $\vec{u}$ be in H, so $\vec{u}$ is in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Then $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ for some scalars c_1, c_2, c_3.

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 - 2\vec{v}_2)$$

$$= (c_1 + c_3)\vec{v}_1 + (c_2 - 2c_3)\vec{v}_2$$

So $\vec{u}$ is in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Since this holds for any vector in H, $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Theorem:

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in a vector space V and $H =$Span(S).

(a.) If one of the vectors in S, say $\mathbf{v}_k$ is a linear combination of the other vectors in S, then the subset of S obtained by eliminating $\mathbf{v}_k$ still spans H.

(b) If $H \neq \{\mathbf{0}\}$, then some subset of S is a basis for H.

If we start with a spanning set, we can eliminate *duplication* and arrive at a basis.

Column Space

Find a basis for the column space matrix B that is in reduced row echelon form

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. = [\vec{b}_1 \;\; \vec{b}_2 \;\; \vec{b}_3 \;\; \vec{b}_4 \;\; \vec{b}_5]$$

The pivot columns are 1, 3, and 5.

Note $\vec{b}_2 = 4\vec{b}_1$. and $\vec{b}_4 = 2\vec{b}_1 - 1\vec{b}_3$

$$\text{Col}\,B = \text{Span}\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5\}$$

by definition.

Our theorem says we can exclude $\vec{b}_4$ and $\vec{b}_2$ to get a basis.

A basis for Col B is

$$\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}.$$

Using the rref

Theorem: If $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ and $B = [\mathbf{b}_1 \cdots \mathbf{b}_n]$ are row equivalent matrices, then Nul $A =$ Nul B. That is, the equations

$$A\mathbf{x} = \mathbf{0} \quad \text{and} \quad B\mathbf{x} = \mathbf{0}$$

have the same solution set.

Note what this means! It means that $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ have **exactly the same linear dependence relationships**!

Theorem:

The pivot columns of a matrix A form a basis of Col A.

Caveat: This means we can use row reduction to identify a basis, but the vectors we obtain will be from the original matrix A. (As illustrated in the following example.)

Find a basis for Col A[1]

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}.$$

We need to identify the pivot columns.

$$\text{rref } A = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1, 3, and 5 are pivot columns.

A basis for Col A is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$$

[1] Use a calculator to do the row reduction.

Find bases for Nul A and Col A

We can use the rref for both.

$$A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 2 & 1 & 5 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 5 \end{bmatrix}$$

Pivot columns are 1 and 2. A basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

For the null space, Consider $A\vec{x} = \vec{0}$.

From the rref

$$x_1 = -3x_3 + 2x_4$$

$$x_2 = x_3 - 5x_4$$

x_3, x_4 are free

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 + 2x_4 \\ x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

A basis for Nul A is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example

Let $H = \{\mathbf{p} \text{ in } \mathbb{P}_2 \mid \mathbf{p}(-1) = 0\}$. Find a basis for H.

A vector in H looks like

$\vec{p}(t) = p_0 + p_1 t + p_2 t^2$ such that

$\vec{p}(-1) = p_0 + p_1(-1) + p_2(-1)^2 = 0$

$$p_0 - p_1 + p_2 = 0$$

we require $p_0 = p_1 - p_2$

with p_1, p_2 - free

Take the example $p_1 = 1$, $p_2 = 0$. Then

$p_0 = 1$ and $\vec{p}(t) = 1 - 1t + 0t^2 = 1 - t$

Letting $p_1 = 0$ and $p_2 = 1$, we get $p_0 = -1$

so $\vec{p}(t) = -1 + 0t + 1t^2 = -1 + t^2$

A basis for H is $\{1-t, -1+t^2\}$.