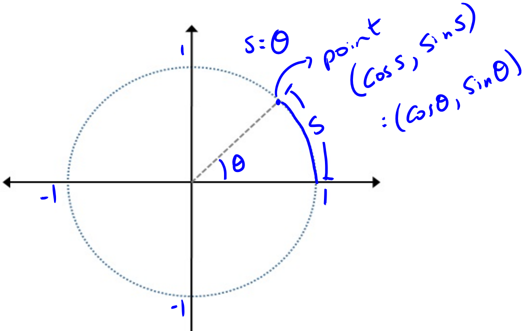
March 11 MATH 1112 sec. 54 Spring 2019

Section 6.5: Trigonometric Functions of a Real Variable and Their Graphs

Let s be **any real number.** Then we can equate an arc of the unit circle with s and consider the point (x, y) determined by s. Since the radius $r = 1$, the angle $\theta = s$. Hence

$$\sin s = y, \quad \cos s = x, \quad \text{and} \quad \tan s = \frac{y}{x}, \text{ when } x \neq 0.$$



Properties of Sine and Cosine

We can deduce some properties from the unit circle interpretation. One property is **periodicity**.

Definition: A function f is said to be **periodic** if there exists a positive constant p such that

$$f(x+p) = f(x)$$

for every x in the domain of f. The smallest such number p is called the **fundamental period** of the function f.

Recall that $f(x+p)$ corresponds to a horizontal shift—p units to the left for $p > 0$. Since $f(x+p) = f(x)$, the shifted graph must be indistinguishable from the unshifted graph.

Graph of a Periodic Function

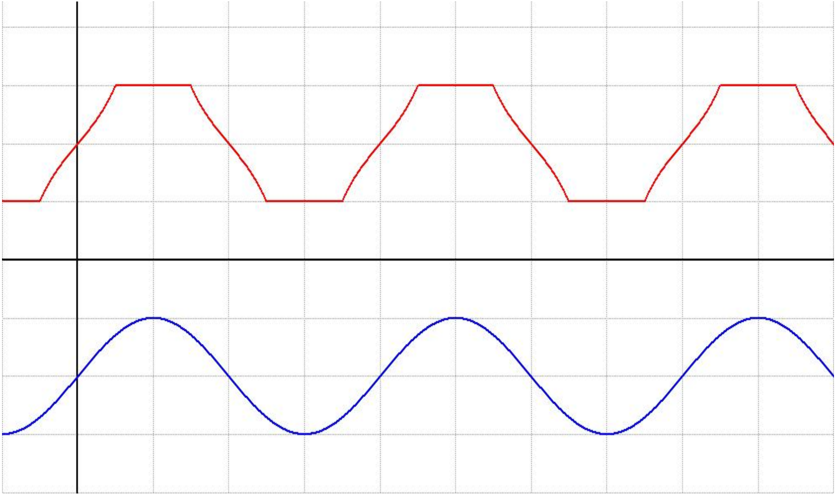
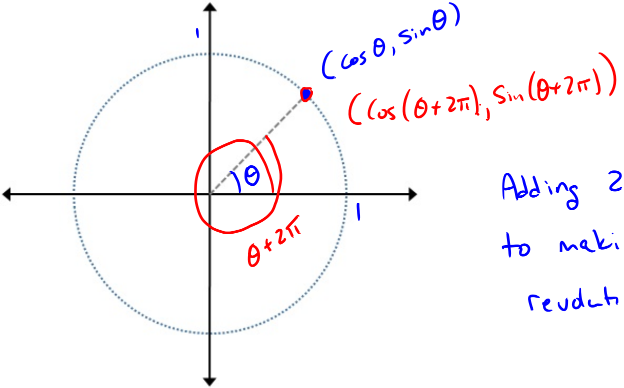


Figure: The profile of a periodic function repeats every p units.

Periodicity of Sine and Cosine

The sine and cosine function are periodic with fundamental period 2π. That is

$$\cos(s + 2\pi) = \cos s \quad \textbf{and} \quad \sin(s + 2\pi) = \sin s \quad \textbf{for all real } s.$$



Adding 2π corresponds to making one full revolution

Domain and Range and Amplitude

Every real number can be equated with a length of an arc (positive in the counter clockwise direction, negative in the clockwise). Hence

Domain: The domain of the sine function is **all real numbers**, and the domain of the cosine function is **all real numbers**.

Domain and Range and Amplitude

Every point (x, y) on the unit circle is such that $|x| \leq 1$ and $|y| \leq 1$. Hence

Range: The range of the sine function is $[-1, 1]$ and the range of the cosine function is also $[-1, 1]$.

The inequalities $|\sin x| \leq 1$ and $|\cos x| \leq 1$ are frequently used in mathematics.

For real number x

$-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$

Question

Suppose for some angle θ that $\tan\theta = \dfrac{5}{4}$.

$\tan\theta = \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta}$

Then $\sin\theta = 5$ and $\cos\theta = 4$.

$-1 \leq \sin\theta \leq 1$

but $5 > 1$

(a) That must be true.

(b) That might be true.

similarly $4 > 1$

(c) That cannot be true.

It is true that $\sin\theta = \dfrac{5}{\sqrt{41}}$ or $\sin\theta = \dfrac{-5}{\sqrt{41}}$

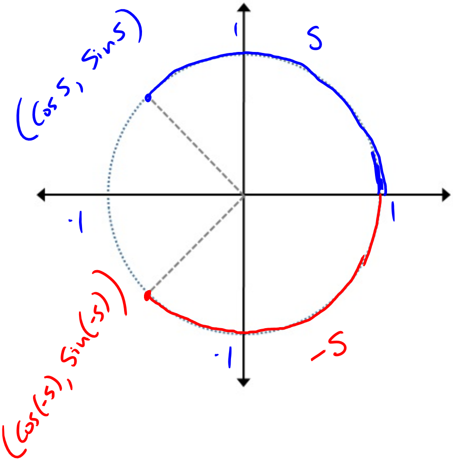
Domain and Range and Amplitude

Definition (Amplitude): The sine and cosine functions oscillate between their maximum and minimum values. Half of the distance between the maximum and minimum is called the **Amplitude**.

$\boxed{\text{The amplitude of both } f(x) = \sin x \text{ and } g(x) = \cos x \text{ is } 1.}$

Symmetry

Consider any real number s and its opposite $-s$.



Note that the x-coordinates are the same

$\cos s = \cos(-s)$ even

The y-coordinates have the same absolute value but opposite signs

$\sin(-s) = -\sin s$ odd

Symmetry of Sine and Cosine

Cosine is Even. The cosine function is an even function. Hence

$$\cos(-s) = \cos s \quad \text{for all real } \; s.$$

Sine is Odd. The sine function is an odd function. Hence

$$\sin(-s) = -\sin s \quad \text{for all real } \; s.$$

Graphs of Sine and Cosine

Due to periodicity and making use of symmetry, we can determine what the entire graphs of sine and cosine look like from the graphs on the interval $0 \leq s < 2\pi$.

Here is an applet to plot one period of the functions $f(s) = \sin s$ and $f(s) = \cos s$.

GeoGebra Graph Applet: Sine and Cosine

Plot of $y = f(x) = \sin x$

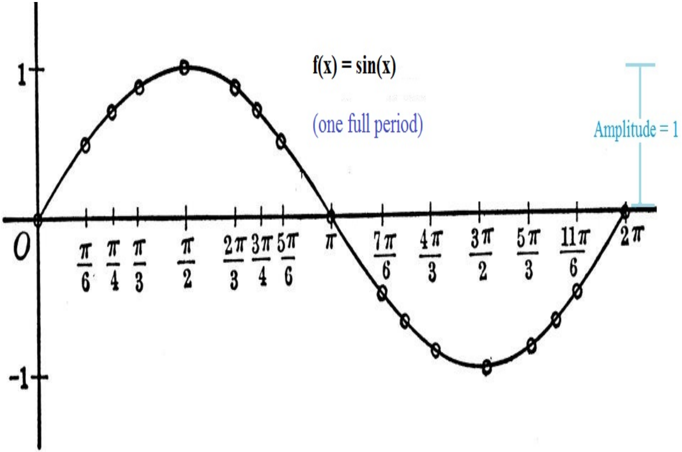


Figure: It's worth noting that major features (maximum, minimum, x-intercepts) divide the period into four equal parts.

Plot of $y = f(x) = \sin x$

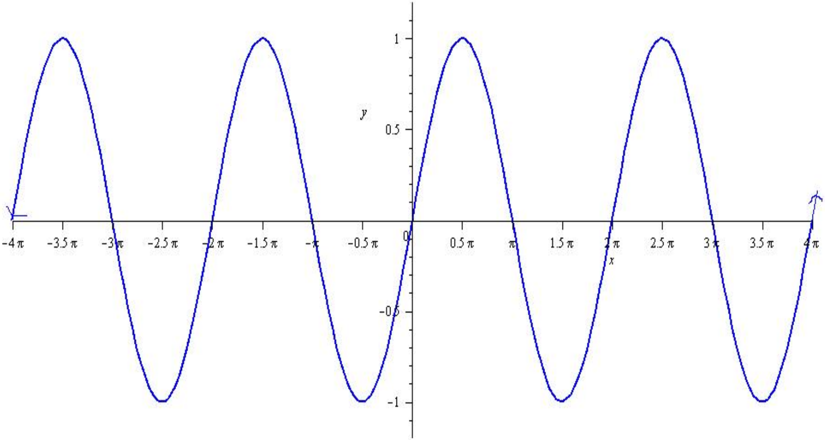


Figure: $y = \sin x$ over four full periods. Note the periodicity and the odd symmetry.

Plot of $y = f(x) = \cos x$

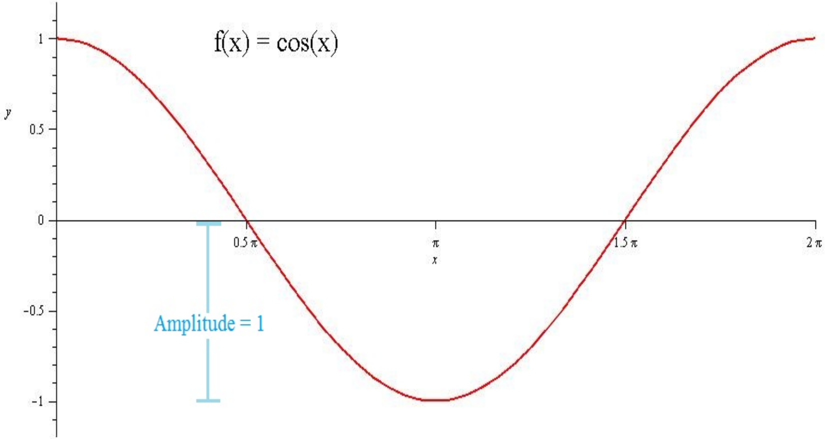


Figure: $y = \cos x$ over one full period.

Plot of $y = f(x) = \cos x$

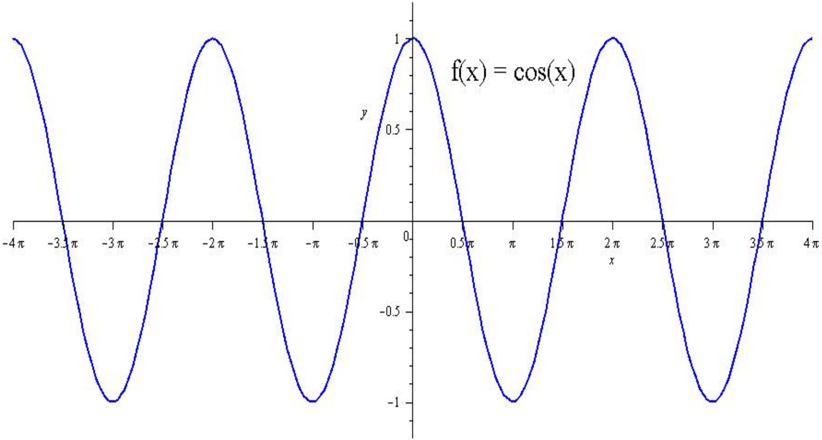


Figure: $y = \cos x$ over four full periods. Note the periodicity and the even symmetry

Plot of $f(x) = \cos x$ and $g(x) = \sin x$ Together

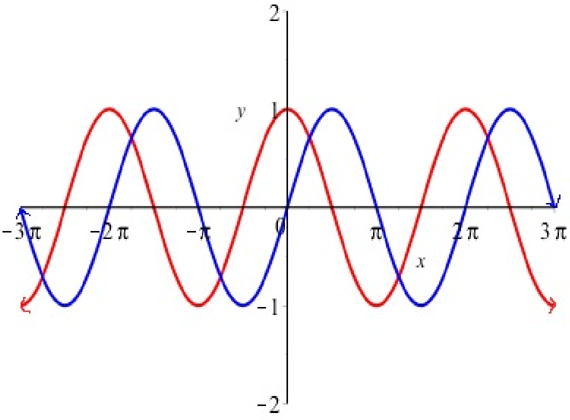


Figure: Note that the cofunction property $\cos(x) = \sin(\pi/2 - x)$ can be seen in the graphs.

A word of caution on using the unit circle

Our goal is to know and understand the properties of the trigonometric functions as **real valued functions of real variables**.

- They appear in many models of periodic processes (sound/water waves, seasonal processes, predator-prey interactions) that have no connection to triangles or circles.

- Right triangles and the unit circle provide powerful mnemonic devices.

- You'll often see x used as the independent variable—in this class and in Calculus to be sure! The point $(x, y) = (\cos s, \sin s)$ on the unit circle are not the same (x, y) as in $y = \cos x$. Get used to it!

Summary of Sine and Cosine Properties

Let $f(x) = \sin x$ and $g(x) = \cos x$. Then

- Both f and g are periodic with fundamental period 2π.
- The domain of both f and g is $(-\infty, \infty)$.
- The range of both f and g is $[-1, 1]$; they have amplitude 1.
- f is an odd function, i.e. $f(-x) = \sin(-x) = -\sin x = -f(x)$.
- g is an even function, i.e. $g(-x) = \cos(-x) = \cos x = g(x)$.
- Both f and g are continuous on their entire domain $(-\infty, \infty)$.
- The zeros of f are integer multiples of π, i.e. $f(n\pi) = 0$ for $n = 0, \pm 1, \pm 2, \ldots$
- The zeros of g are odd integer multiples of $\pi/2$, $g\left(\frac{m\pi}{2}\right) = 0$ for $m = \pm 1, \pm 3, \ldots$

Question

Let $f(x) = \sin x$, and suppose that $f(a) = -0.62$ for some real number a. Which of the following is true?

(a) $f(a + 2\pi) = -0.62$ ✓

(b) $f(2a) = -1.24$ ✗

(c) $f(-a) = 0.62$ ✓

(d) Only (a) and (b) are true.

(e) Only (a) and (c) are true.

$f(x+2\pi) = f(x)$ for all x

Sine is odd so

$f(-a) = -f(a)$

$= -(-0.62) = 0.62$

$-1 \leq \sin(2a) \leq 1$

Question

True or False: The equation $\cos x = 2$ has no real solutions.

(a) True, and I'm confident.

(b) True, but I'm not confident.

(c) False, and I'm confident.

(d) False, but I'm not confident.