March 11 Math 2306 sec. 54 Spring 2019

Section 10: Variation of Parameters

For the second order equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation.

Variation of parameters provides a particular solution in the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x)$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where

$$u_1 = \int \frac{-g(x)y_2(x)}{W}\,dx \quad \text{and} \quad u_2 = \int \frac{g(x)y_1(x)}{W}\,dx$$

and $W = W(y_1, y_2)(x)$ is the Wronskian of the solutions of the associated homogeneous equation.

Example:

Solve the ODE $y'' + y = \tan x$.

We need y_c, the solution to $y'' + y = 0$. The characteristic equation is $m^2 + 1 = 0$. Solve

$$m^2 = -1 \Rightarrow m = \pm\sqrt{-1} = \pm i, \quad \alpha = 0, \beta = 1$$

$$y_1 = \cos x, \quad y_2 = \sin x.$$

The ODE is in standard form, so $g(x) = \tan x$. We need W.

$$W(y_1, y_2)(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = 1$$

$y_p = u_1 y_1 + u_2 y_2$ where

$$u_1 = \int \frac{-g(x)\, y_2(x)}{W}\, dx \qquad u_2 = \int \frac{g(x)\, y_1(x)}{W}\, dx$$

$$= \int \frac{-\tan x \;\sin x}{1}\, dx \qquad = \int \frac{\tan x \;\cos x}{1}\, dx$$

Finding u_2

$$u_2 = \int \tan x \,\cos x \, dx = \int \sin x \, dx = -\cos x$$

$$u_1 = \int -\tan x \,\sin x \, dx = -\int \frac{\sin^2 x}{\cos x}\, dx$$

$$= -\int \frac{(1-\cos^2 x)}{\cos x}\,dx = \int (\cos x - \sec x)\,dx$$

$$= \sin x - \ln|\sec x + \tan x|$$

$$y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

$$= \left(\sin x - \ln|\sec x + \tan x|\right)\cos x + (-\cos x)\sin x$$

$$= \sin x\cos x - \cos x\ln|\sec x + \tan x| - \cos x\sin x$$

$$= -\cos x\ln|\sec x + \tan x|$$

The general solution is

$$y = C_1 \cos x + C_2 \sin x - \cos x \ln\left|\sec x + \tan x\right|$$

Example:

Solve the ODE

$$x^2y'' + xy' - 4y = \ln x,$$

given that $y_c = c_1x^2 + c_2x^{-2}$ is the complementary solution.

From the given y_c, we have $y_1 = x^2$, $y_2 = x^{-2}$.

We need the ODE in standard form to identify $g(x)$.

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2} \qquad g(x) = \frac{\ln x}{x^2}.$$

The Wronskian

$$W(y_1,y_2)(x) = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^{-1} - 2x^{-1} = -4x^{-1}$$

Then $y_p = u_1 y_1 + u_2 y_2$ where

$$u_1 = \int \frac{-g(x)\, y_2(x)}{W}\, dx \qquad u_2 = \int \frac{g(x)\, y_1(x)}{W}\, dx$$

$$= \int \frac{-\left(\frac{\ln x}{x^2}\right) x^{-2}}{-4x^{-1}}\, dx \qquad u_2 = \int \frac{\left(\frac{\ln x}{x^2}\right) x^2}{-4x^{-1}}\, dx$$

$$u_1 = \int \frac{1}{4} (\ln x)\, x^{-4} \cdot x\, dx = \int \frac{1}{4} (\ln x)\, x^{-3}\, dx$$

Integrate by parts with $u = \ln x$, $dv = x^{-3}\, dx$

$du = \frac{1}{x}\, dx$, $v = \frac{x^{-2}}{-2}$

$$= \frac{1}{4}\left(\frac{-1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2}\right) = \frac{-1}{8}x^{-2}\ln x - \frac{1}{16}x^{-2}$$

$$u_2 = \int \frac{-1}{4}(\ln x)\, x^{-2}\cdot x^{2}\cdot x\, dx = \int \frac{-1}{4}\, x\ln x\; dx$$

Integrate by parts with $u = \ln x$, $dv = x\,dx$

$du = \frac{1}{x}dx$, $v = \frac{x^2}{2}$

$$= \frac{-1}{4}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right) = \frac{-1}{8}x^2\ln x + \frac{1}{16}x^2$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(-\frac{1}{8} x^{-2} \ln x - \frac{1}{16} x^{-2}\right) x^2 + \left(-\frac{1}{8} x^2 \ln x + \frac{1}{16} x^2\right) x^{-2}$$

$$= -\frac{1}{8} \ln x - \frac{1}{16} - \frac{1}{8} \ln x + \frac{1}{16}$$

$$= -\frac{1}{4} \ln x$$

The general solution is

$$y = c_1 x^2 + c_2 x^{-2} - \frac{1}{4} \ln x$$

Let's verify that our y_p is correct. Sub it into

$$x^2 y'' + xy' - 4y = \ln x$$

$$y_p = \frac{-1}{4}\ln x, \quad y_p' = \frac{-1}{4}\frac{1}{x} \quad y_p'' = \frac{1}{4}\left(\frac{1}{x^2}\right)$$

$$x^2 y_p'' + x y_p' - 4y_p =$$

$$x^2\left(\frac{1}{4}\frac{1}{x^2}\right) + x\left(\frac{-1}{4}\frac{1}{x}\right) - 4\left(\frac{-1}{4}\ln x\right) =$$

$$\frac{1}{4} - \frac{1}{4} + \ln x = \ln x$$

$\ln x = \ln x$ ✓

Solve the IVP

$$x^2y'' + xy' - 4y = \ln x, \quad y(1) = -1, \quad y'(1) = 0$$

The general solution is $\quad y = c_1 x^2 + c_2 x^{-2} - \frac{1}{4}\ln x$

Apply the I.C. $\quad y' = 2c_1 x - 2c_2 x^{-3} - \frac{1}{4x}$

$$y(1) = c_1(1^2) + c_2(1^{-2}) - \frac{1}{4}\ln 1 = -1$$

$$c_1 + c_2 = -1$$

$$y'(1) = 2c_1(1) - 2c_2(1^{-3}) - \frac{1}{4(1)} = 0$$

$$2c_1 - 2c_2 = \frac{1}{4}$$

Solving the system

$$2c_1 + 2c_2 = -2$$
$$2c_1 - 2c_2 = \frac{1}{4}$$

add $\quad 4c_1 = \frac{-7}{4} \qquad c_1 = \frac{-7}{16}$

subtract $\quad 4c_2 = \frac{-9}{4} \qquad c_2 = \frac{-9}{16}$

The solution to the IVP is

$$\boxed{y = \frac{-7}{16}x^2 - \frac{9}{16}x^{-2} - \frac{1}{4}\ln x}$$