March 13 MATH 1112 sec. 54 Spring 2019

Section 6.5: Trigonometric Functions of a Real Variable and Their Graphs

Let $f(x) = \sin x$ and $g(x) = \cos x$. Then

- Both f and g are periodic with fundamental period 2π.
- The domain of both f and g is $(-\infty, \infty)$.
- The range of both f and g is $[-1, 1]$; they have amplitude 1.
- f is an odd function, i.e. $f(-x) = \sin(-x) = -\sin x = -f(x)$.
- g is an even function, i.e. $g(-x) = \cos(-x) = \cos x = g(x)$.
- Both f and g are continuous on their entire domain $(-\infty, \infty)$.
- The zeros of f are integer multiples of π, i.e. $f(n\pi) = 0$ for $n = 0, \pm 1, \pm 2, \ldots$
- The zeros of g are odd integer multiples of $\pi/2$, $g\left(\frac{m\pi}{2}\right) = 0$ for $m = \pm 1, \pm 3, \ldots$

Plot of $f(x) = \cos x$ and $g(x) = \sin x$ Together

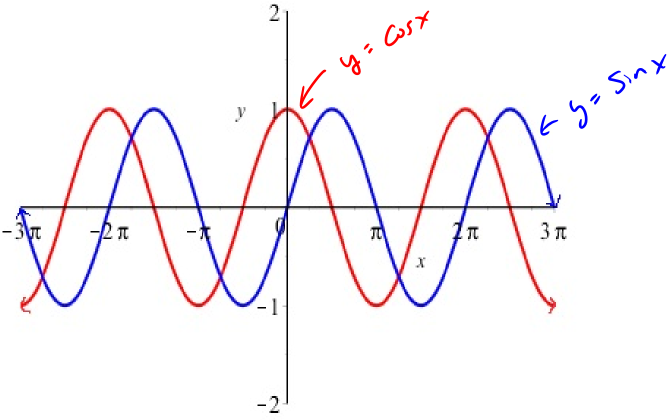

Figure: Note that the cofunction property $\cos(x) = \sin(\pi/2 - x)$ can be seen in the graphs.

The Tangent

The function $\tan s = \dfrac{\sin s}{\cos s}$. Recall that

$$\cos s = 0 \quad \text{whenever} \quad s = \frac{m\pi}{2} \quad \text{for} \quad m = \pm 1, \pm 3, \pm 5, \dots$$

When $\cos s = 0$, $\sin s$ is either 1 or -1. Hence

Domain: The domain of the tangent function is all real number **except** odd multiples of $\pi/2$. We can write this as

$$\left\{ s \;\middle|\; s \neq \frac{\pi}{2} + k\pi,\ k = 0, \pm 1, \pm 2, \dots \right\}$$

Moreover, the graph of the tangent function has vertical asymptotes at each odd multiple of $\pi/2$.

The Tangent

Range: The range of the tangent function is **all real numbers.**

Symmetry: The function $f(s) = \tan s$ is odd. That is

$$f(-s) = \tan(-s) = -\tan s = -f(s).$$

Perodicity: The tangent function is periodic with fundamental period π. That is

$$\tan(s + \pi) = \tan s \quad \text{for all } s \text{ in the domain.}$$

Note: The period of the tangent function is π. This is different from the period of the sine and cosine.

Minute Exercise

We have the *nice* sine and cosine values

s	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin s$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

and

s	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos s$	$\mathbf{1}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

Take a moment and fill in the table of tangent values (as possible) using $\tan s = \sin s / \cos s$.

s	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan s$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

The Tangent

A few key tangent values:

s	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan s$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef.

And due to symmetry

s	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0
$\tan s$	undef.	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

Here is an applet to plot two periods of the function $f(s) = \tan s$.

GeoGebra Graph Applet: Sine, Cosine, and Tangent

Basic Plot $f(x) = \tan x$

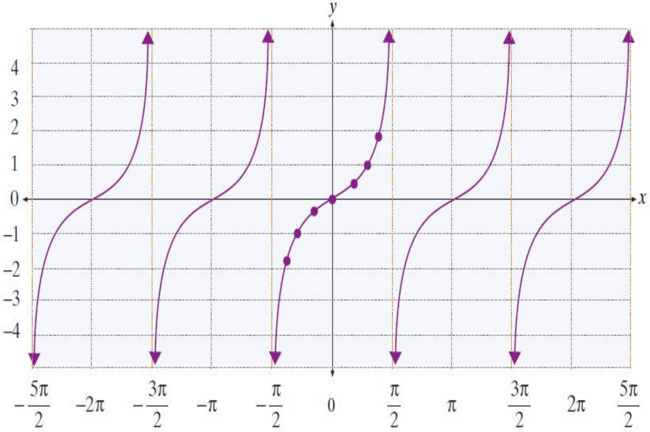

Figure: Plot of several periods of $f(x) = \tan x$. Note that the interval between adjacent asymptotes is the period π.

Cotangent: Using the Cofunction ID and Symmetry

$$\cot s = \tan\left(\frac{\pi}{2} - s\right) = \tan\left(-\left(s - \frac{\pi}{2}\right)\right) = -\tan\left(s - \frac{\pi}{2}\right).$$

So the graph of $f(s) = \cot s$ is the graph of $g(s) = \tan s$ under a horizontal shift $\pi/2$ units to the right followed by a reflection in the s-axis.

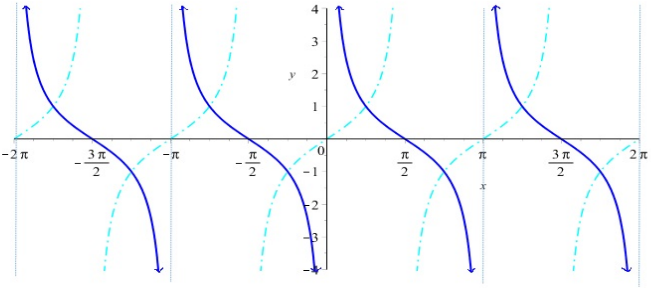

Figure: Plot of $f(x) = \cot x$. Note that the lines $x = n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$ are vertical asymptotes to the graph. The dashed curve is $y = \tan x$.

Cosecant and Secant

Domains: Since $\sin(n\pi) = 0$ for integers n,

$$\textbf{Domain}(\csc s) \;=\; \{s \,|\, s \neq n\pi, \text{ for integers } n\}.$$

Since $\cos\left(\frac{\pi}{2} + n\pi\right) = 0$ for integers n, the domain of $\sec s$ is

$$\textbf{Domain}(\sec s) \;=\; \left\{s \,\middle|\, s \neq \frac{\pi}{2} + n\pi, \text{ for integers } n\right\}.$$

Ranges: Note that Recall $|\sin s| \leq 1$ and $|\cos s| \leq 1$

$$|\csc s| = \frac{1}{|\sin s|} \geq 1 \quad \text{and} \quad |\sec s| = \frac{1}{|\cos s|} \geq 1$$

so the range of both $\csc s$ and $\sec s$ is

$$(-\infty, -1] \cup [1, \infty).$$

Cosecant: Using $\csc s = \frac{1}{\sin s}$

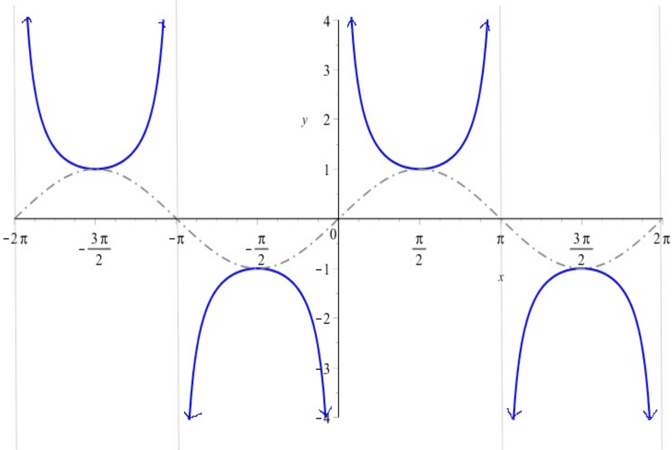

Figure: Two periods of $f(s) = \csc s$. The dashed curve is $y = \sin s$. Note the asymptotes $s = n\pi$ for integers n where $\sin s$ takes its zeros. The curves meet at the relative extrema and have the same period 2π.

Secant: Using $\sec s = \dfrac{1}{\cos s}$

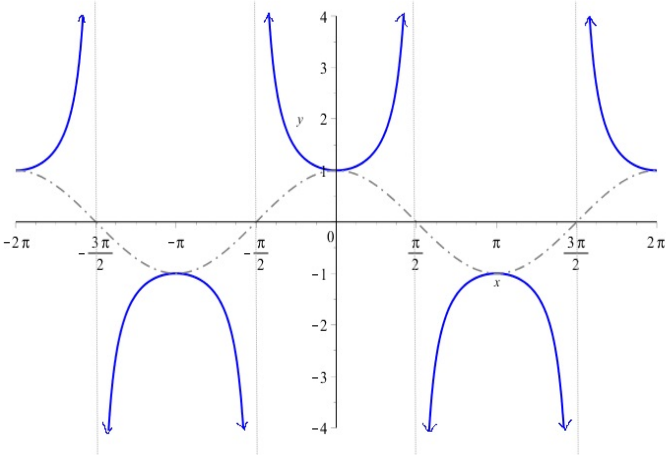

Figure: Two periods of $f(s) = \sec s$. The dashed curve is $y = \cos s$. Note the asymptotes $s = \pi/2 + n\pi$ for integers n where $\cos s$ takes its zeros. The curves meet at the relative extrema and have the same period 2π.

Minute Exercise

We have the symmetry for sine, cosine, and tangent:
$\sin(-x) = -\sin x \ \ \cos(-x) = \cos x$, $\tan(-x) = -\tan x$.

We have the reciprocal relationships

$$\csc x = \frac{1}{\sin x} \quad \sec x = \frac{1}{\cos x} \quad \cot x = \frac{1}{\tan x}.$$

From these, take a moment and deduce the symmetry (even or odd) for the remaining three functions.

- $\csc x$ is odd so $\csc(-x) =$ $-\csc(x)$
- $\sec x$ is even so $\sec(-x) =$ $\sec(x)$
- $\cot x$ is odd so $\cot(-x) =$ $-\cot(x)$