# March 15 Math 2306 sec. 53 Spring 2019

## Section 11: Linear Mechanical Equations

**Simple Harmonic Motion:** In the absence of any damping or external driving force, we determined the displacement $x$ from equilibrium of an object suspended from a spring according to Hooke's law:

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, $x_0$ and $x_1$ are the initial position (relative to equilibrium) and velocity, respectively. The value

$$\omega^2 = \frac{k}{m}$$

where $k$ is the spring constant and $m$ the mass of the suspended object.

# The equation of motion

The solution to the IVP

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

is called the **equation of motion.**

We took the sign convention that the direction up is positive ($x > 0$) and down is negative ($x < 0$).

# Free Damped Motion

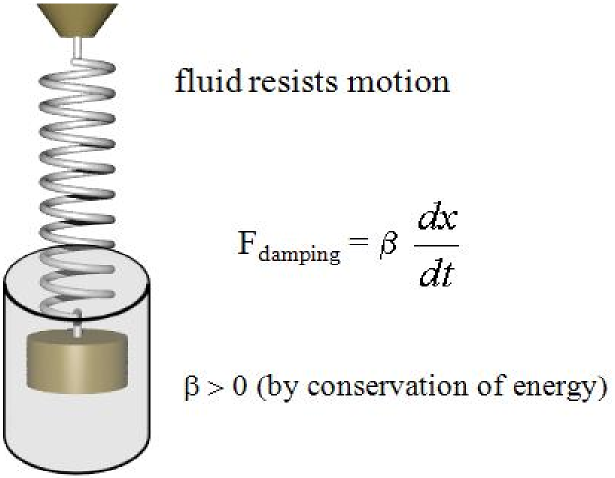


Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

# Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of damping + Force of spring

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

# Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

2 distinct real roots

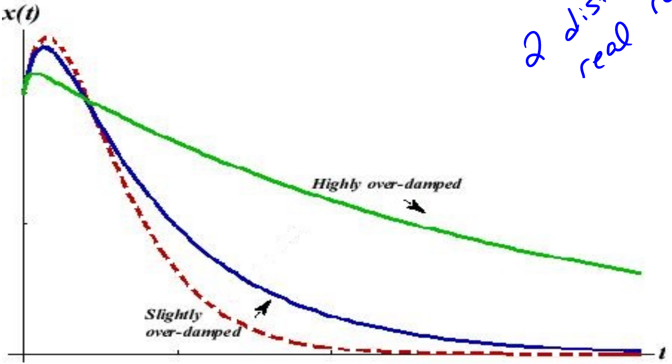


Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

# Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

1 repeated real root

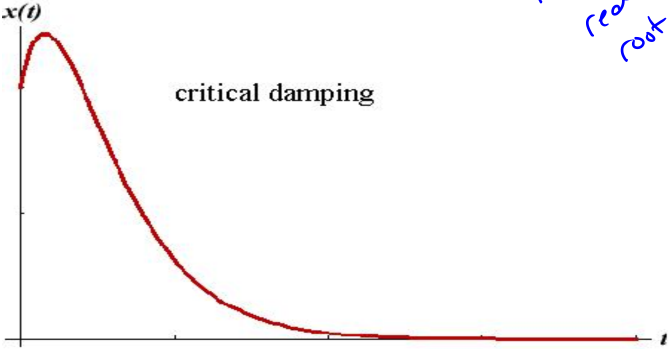


Figure: One real root. No oscillations. Fastest approach to equilibrium.

# Case 3: $\lambda^2 < \omega^2$ Underdamped

Complex conjugate roots

$$x(t) = e^{-\lambda t}\left(c_1\cos(\omega_1 t) + c_2\sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

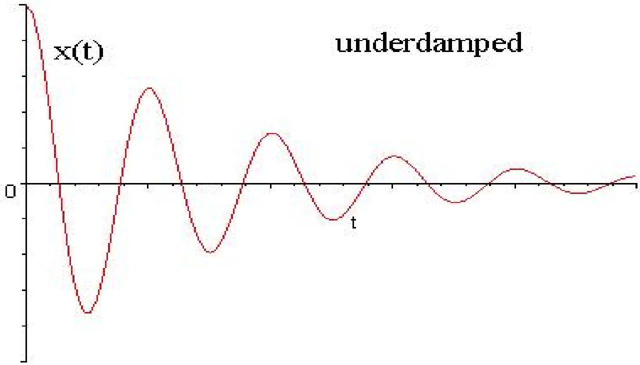


Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

# Comparison of Damping

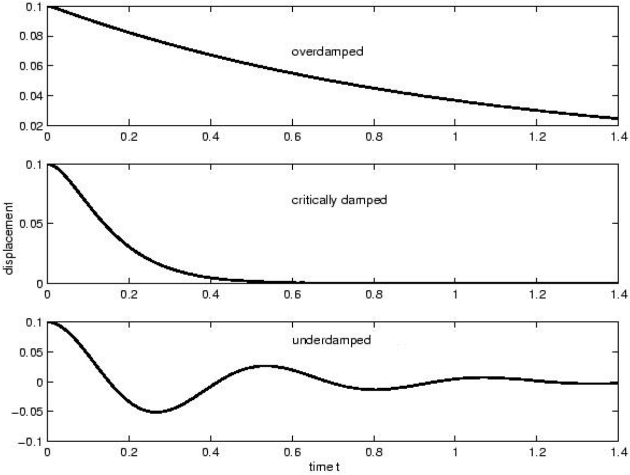


Figure: Comparison of motion for the three damping types.

# Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The displacement $x$ satisfies the ODE

$mx'' + \beta x' + kx = 0$. From the statement

$m = 2$, $k = 12$, $\beta = 10$. The ODE is

$$2x'' + 10x' + 12x = 0$$

In standard form $\quad x'' + 5x' + 6x = 0$.

The characteristic equation (using $r$)

$$r^2 + 5r + 6 = 0$$

$$(r+2)(r+3) = 0 \implies r = -2 \text{ or } r = -3$$

There are two distinct real roots,

the system is overdamped.

Note here, that $\omega^2 = \frac{k}{m} = \frac{12}{2} = 6$ and

$$2\lambda = \frac{\beta}{m} = \frac{10}{2} = 5 \implies \lambda = \frac{5}{2}.$$

Then $\lambda^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$ and $\omega^2 = 6 = \frac{24}{4}$

so $\lambda^2 > \omega^2$ as expected

for an overdamped system.

# Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

The displacement $x$ solves

$$mx'' + \beta x' + kx = 0.$$

Here $m = 3$, $k = 12$, $\beta = 12$. The ODE is

$$3x'' + 12x' + 12x = 0$$

i.e. $$x'' + 4x' + 4x = 0$$

The characteristic equation is

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \implies r = -2 \text{ repeated}$$

$\boxed{\text{The motion is critically damped}}$

Now we solve the IVP

$$x'' + 4x' + 4x = 0 \quad \text{subject to}$$

$$x(0) = 0 \qquad x'(0) = 1$$

released from equilibrium

initial upward velocity of 1 m/sec

The general solution to the ODE is

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

Apply the I.C.

$$x'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$x(0) = c_1 e^0 + c_2 \cdot 0 e^0 = 0 \implies c_1 = 0$$

$$x'(0) = c_2 e^0 - 2c_2 \cdot 0 e^0 = 1 \implies c_2 = 1$$

The displacement $x$ for $t > 0$ is

$$x(t) = t e^{-2t}$$

# Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out $m$ and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$