March 15 Math 2306 sec. 54 Spring 2019

Section 11: Linear Mechanical Equations

Simple Harmonic Motion: In the absence of any damping or external driving force, we determined the displacement x from equilibrium of an object suspended from a spring according to Hooke's law:

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The value

$$\omega^2 = \frac{k}{m}$$

where k is the spring constant and m the mass of the suspended object.

The equation of motion

The solution to the IVP

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

is called the **equation of motion.**

We took the sign convention that the direction up is positive ($x > 0$) and down is negative ($x < 0$).

Free Damped Motion

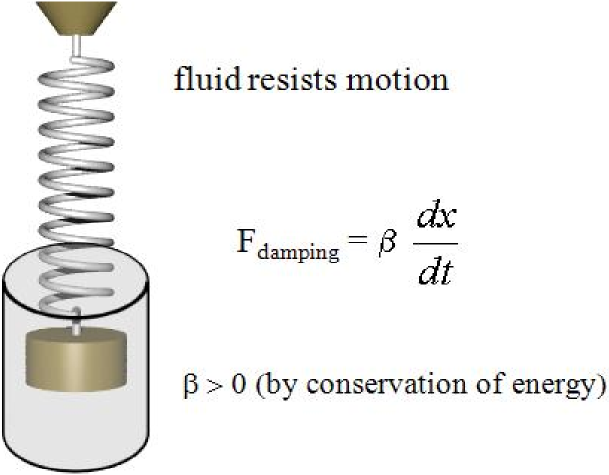

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of damping + Force of spring

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

2 distinct real roots

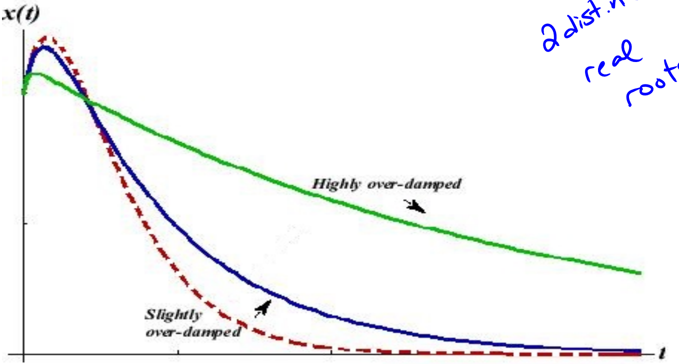

Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

1 repeated real root

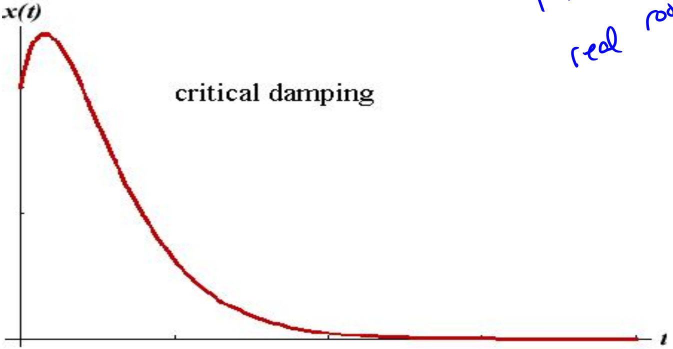

Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

complex conjugate roots

$$x(t) = e^{-\lambda t}\left(c_1\cos(\omega_1 t) + c_2\sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

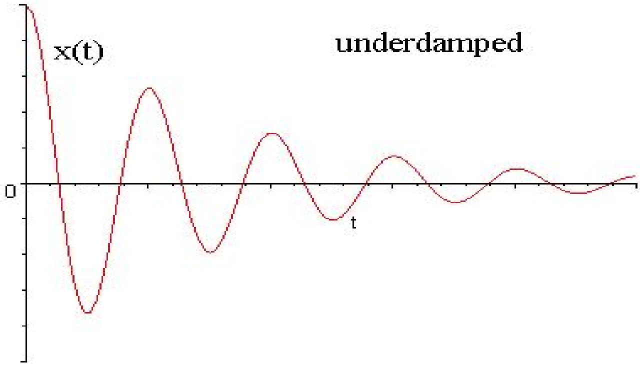

Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping

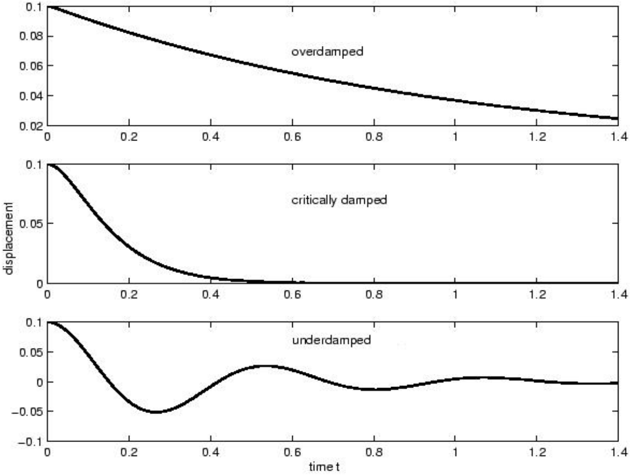

Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The displacement x satisfies

$$mx'' + \beta x' + kx = 0.$$

Here, $m = 2$, $k = 12$, and $\beta = 10$. The ODE is

$$2x'' + 10x' + 12x = 0.$$

In standard form $x'' + 5x' + 6x = 0$

Using r for the characteristic equation, we have

$$r^2 + 5r + 6 = 0.$$

$$(r+2)(r+3) = 0 \implies r = -2 \text{ or } r = -3.$$

Two distinct real roots; the system is overdamped.

Note that $\omega^2 = \frac{k}{m} = \frac{12}{2} = 6$. And

$$2\lambda = \frac{\beta}{m} = \frac{10}{2} = 5 \implies \lambda = \frac{5}{2}.$$

So $\lambda^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$ and $\omega^2 = 6 = \frac{24}{4}$

as expected $\lambda^2 > \omega^2$

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

The displacement satisfies

$$mx'' + \beta x' + kx = 0 \quad \text{where}$$

$m = 3$, $k = 12$, $\beta = 12$. The ODE is

$$3x'' + 12x' + 12x = 0$$

in standard form $\quad x'' + 4x' + 4x = 0$

The characteristic equation is

$$r^2+4r+4=0$$

$$(r+2)^2=0 \quad \Rightarrow \quad r=-2 \text{ repeated}$$

One repeated root $\Rightarrow$ the motion is critically damped.

The displacement solves

$$x''+4x'+4x=0 \quad \text{subject to}$$

$$x(0)=0 \qquad x'(0)=1$$

released from equilibrium   initial upward velocity of $1\ ^m/sec$

From the characteristic equation

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

Impose the initial conditions.

$$x'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$x(0) = c_1 e^{0} + c_2 \cdot 0 e^{0} = 0 \Rightarrow c_1 = 0$$

$$x'(0) = c_2 e^{0} - 2c_2 \cdot 0 e^{0} = 1 \Rightarrow c_2 = 1$$

The displacement for $t > 0$ is $x(t) = te^{-2t}$.

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$