

Section 11: Linear Mechanical Equations

Simple Harmonic Motion: In the absence of any damping or external driving force, we determined the displacement x from equilibrium of an object suspended from a spring according to Hooke's law:

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \quad (1)$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The value

$$\omega^2 = \frac{k}{m}$$

where k is the spring constant and m the mass of the suspended object.

The equation of motion

The solution to the IVP

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

is called the **equation of motion**.

We took the sign convention that the direction up is positive ($x > 0$) and down is negative ($x < 0$).

Free Damped Motion



fluid resists motion

$$F_{\text{damping}} = \beta \frac{dx}{dt}$$

$\beta > 0$ (by conservation of energy)

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of damping + Force of spring

$$m \frac{d^2 x}{dt^2} = -\beta \frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t} \left(c_1 e^{t\sqrt{\lambda^2 - \omega^2}} + c_2 e^{-t\sqrt{\lambda^2 - \omega^2}} \right)$$

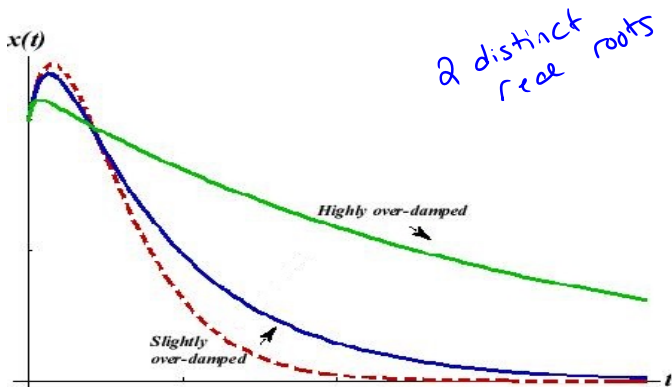


Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t} (c_1 + c_2 t)$$

*1 repeated
real root*

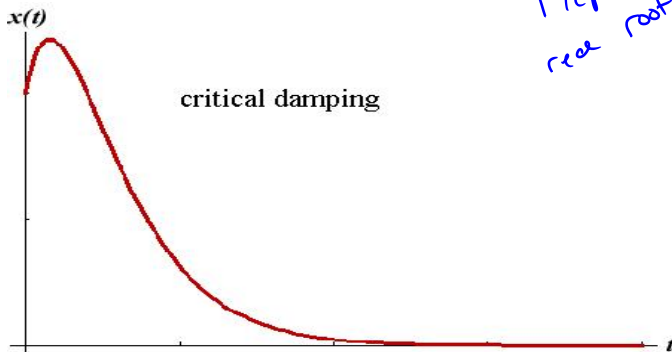


Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

Complex conjugate roots

$$x(t) = e^{-\lambda t} (c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

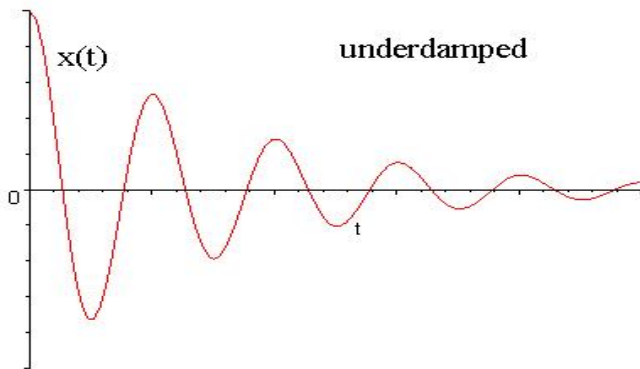


Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping

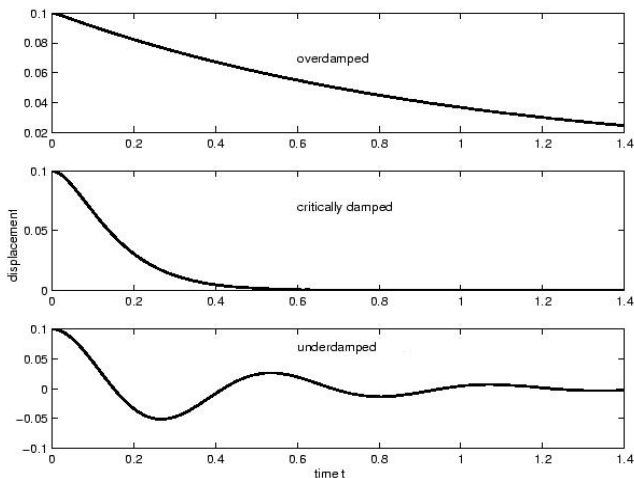


Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The displacement satisfies

$$m x'' + \beta x' + k x = 0$$

From the statement, $m = 2$, $k = 12$, and $\beta = 10$

$$\frac{k m}{\text{sec}^2}, \beta \frac{dx}{dt} = \beta \frac{\text{N}}{\text{sec}} \Rightarrow \beta \sim \frac{\text{kg}}{\text{sec}}$$

Our ODE is

$$2x'' + 10x' + 12x = 0$$

i.e. $x'' + 5x' + 6x = 0$ using r , the

Characteristic equation is

$$r^2 + 5r + 6 = 0$$

$$(r+2)(r+3) = 0 \Rightarrow r = -2 \text{ or } r = -3$$

Two distinct roots means the system is over damped.

Note that $2\lambda = \frac{B}{m} = \frac{10}{2} = 5$ so $\lambda = \frac{5}{2}$.

$$\text{And } \omega^2 = \frac{k}{m} = \frac{12}{2} = 6$$

$$\lambda^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4} \quad \text{and} \quad \omega^2 = 6 = \frac{24}{4}$$

$$\text{so } \lambda^2 > \omega^2$$

confirming the system is
over damped.

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

The ODE is $mx'' + \beta x' + kx = 0$. Here

$m = 3$, $k = 12$, and $\beta = 12$. So

$$3x'' + 12x' + 12x = 0$$

In standard form $x'' + 4x' + 4x = 0$ with

Characteristic equation

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \Rightarrow r = -2 \text{ repeated.}$$

The system is critically damped.

Now we solve the IVP

$$x'' + 4x' + 4x = 0 \text{ subject to}$$

$$x(0) = 0 \quad \text{and} \quad x'(0) = 1$$

released \uparrow
from
equilibrium

\uparrow
initial upward
velocity 1 m/sec.

The general solution based on the characteristic equation is

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

Apply the I.C.

$$x'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$x(0) = c_1 e^0 + c_2 \cdot 0 e^0 = 0 \Rightarrow c_1 = 0$$

$$x'(0) = c_2 e^0 - 2c_2 \cdot 0 e^0 = 1 \Rightarrow c_2 = 1$$

The position for all $t > 0$ is

$$x(t) = te^{-2t}$$

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m \frac{d^2 x}{dt^2} = -\beta \frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = F(t)$$