March 18 MATH 1112 sec. 54 Spring 2019

Section 7.4: Inverse Trigonometric Functions

Question: If someone asks "what is the sine of $\frac{\pi}{6}$?" we can respond with the answer (from memory or perhaps using a calculator) "$\frac{1}{2}$". What if the question is reversed? What if someone asks

"What angle has a sine value of $\frac{1}{2}$?"

One answer is $\frac{\pi}{6}$. Another answer is $\frac{5\pi}{6}$.

There are, in fact, infinitely many correct answers.

Restricting the Domain of sin(x)

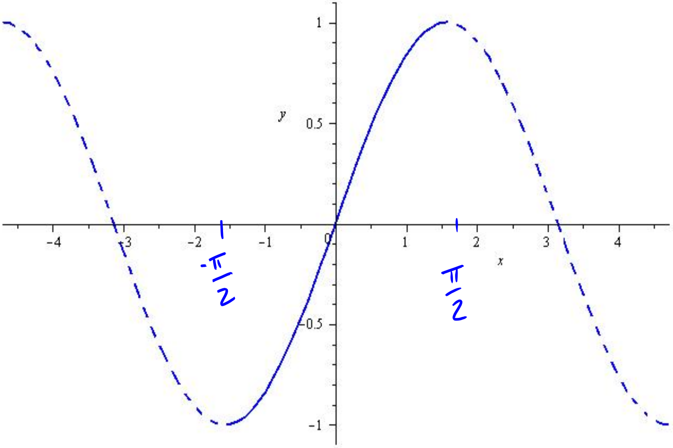


Figure: To define an inverse sine function, we start by restricting the domain of sin(x) to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The Inverse Sine Function (a.k.a. arcsine function)

Definition: For x in the interval $[-1, 1]$ the inverse sine of x is denoted by either

$$\sin^{-1}(x) \quad \text{or} \quad \arcsin(x)$$

and is defined by the relationship

$$y = \sin^{-1}(x) \quad \iff \quad x = \sin(y) \quad \text{where} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

The Domain of the Inverse Sine is $-1 \le x \le 1$.

The Range of the Inverse Sine is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Notation Warning!

Caution: We must remember not to confuse the superscript -1 notation with reciprocal. That is

$$\sin^{-1}(x) \neq \frac{1}{\sin(x)}.$$

If we want to indicate a reciprocal, we should use parentheses or trigonometric identities

$$\frac{1}{\sin(x)} = (\sin(x))^{-1} \quad \text{or write} \quad \frac{1}{\sin(x)} = \csc(x).$$

Conceptual Definition[1]

quadrants IV and I

We can think of the inverse sine function in the following way:

$\sin^{-1}(x)$ is the *angle* between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x.

For example $\sin^{-1}\left(\frac{1}{2}\right)$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is $\frac{1}{2}$.

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

[1]We want to consider $f(x) = \sin^{-1} x$ as a real valued function of a real variable without necessary reference to angles, triangles, or circles. But the above is a **very useful** conceptual device for working with and evaluating this function.

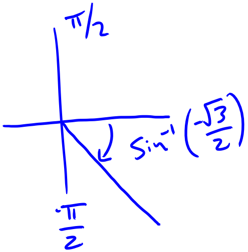
Example

Evaluate exactly.

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$= \frac{-\pi}{3}$

$\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ is the angle between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ whose sine is $\frac{-\sqrt{3}}{2}$



* Note $\frac{-\pi}{2} < \frac{-\pi}{3} < \frac{\pi}{2}$

and $\sin\left(\frac{-\pi}{3}\right) = \frac{-\sqrt{3}}{2}$

Question

The value of $\sin^{-1} 1$ is

(a) $\frac{\pi}{2}$

(b) $\frac{\pi}{2}$ and $\frac{5\pi}{2}$

(c) 0

(d) 0 and π

If $\theta = \sin^{-1}(1)$, then

$\sin(\theta) = 1$

and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

The Graph of the Arcsine

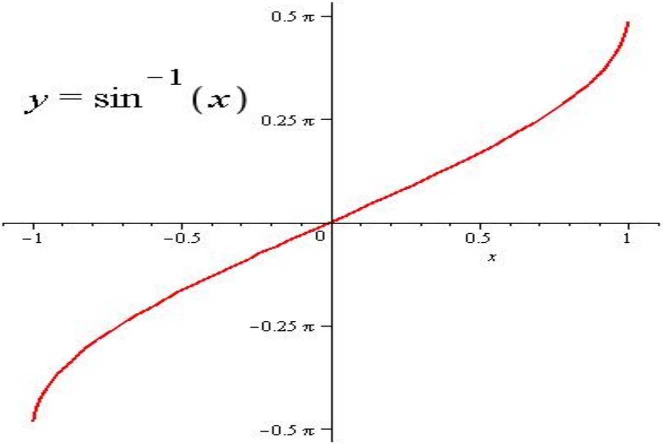


Figure: Note that the domain is $-1 \leq x \leq 1$ and the range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

The Inverse Cosine Function

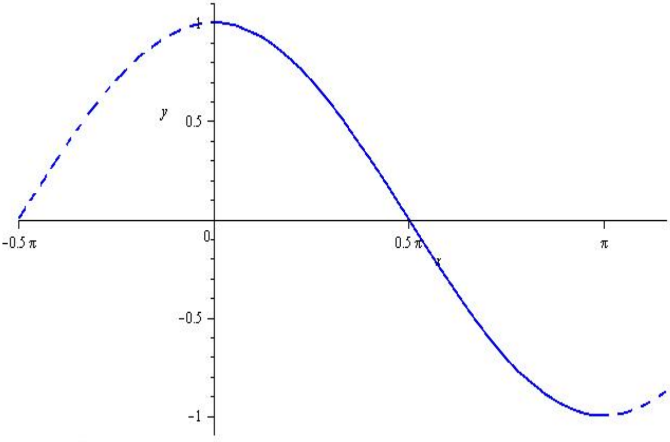


Figure: To define an inverse cosine function, we start by restricting the domain of $\cos(x)$ to the interval $[0, \pi]$

The Inverse Cosine Function (a.k.a. arccosine function)

Definition: For x in the interval $[-1, 1]$ the inverse cosine of x is denoted by either

$$\cos^{-1}(x) \quad \text{or} \quad \arccos(x)$$

and is defined by the relationship

$$y = \cos^{-1}(x) \quad \Longleftrightarrow \quad x = \cos(y) \quad \text{where} \quad 0 \le y \le \pi.$$

quadrant I and II

The Domain of the Inverse Cosine is $-1 \le x \le 1$.

The Range of the Inverse Cosine is $0 \le y \le \pi$.

The Graph of the Arccosine

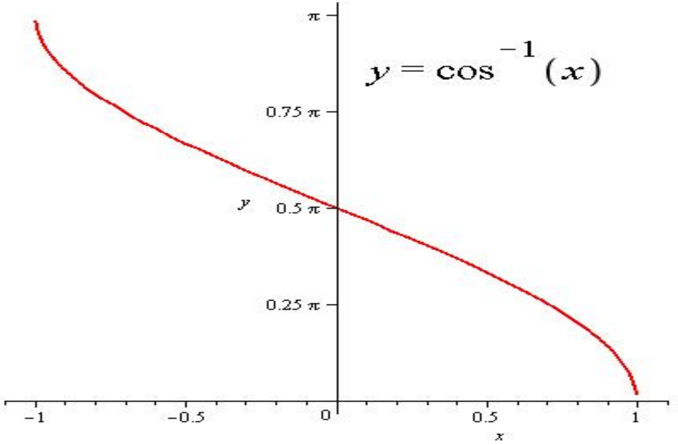


Figure: Note that the domain is $-1 \le x \le 1$ and the range is $0 \le y \le \pi$.

Conceptual Definition

We can think of the inverse cosine function in the following way:

$\cos^{-1}(x)$ is the *angle* between 0 and π whose cosine is x.

$\cos^{-1}\left(\frac{1}{2}\right)$ is the angle between 0 and π whose cosine is $\frac{1}{2}$. $\cos\frac{\pi}{3}=\frac{1}{2}$ and $0<\frac{\pi}{3}<\pi$

So $\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$.

Question

Both of the following are true statements:

$$\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

The value of $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is

(a) $\frac{\pi}{4}$ and $-\frac{\pi}{4}$ ← can't have 2 outputs

(b) just $-\frac{\pi}{4}$ ← not in the range $[0, \pi]$

(c) just $\frac{\pi}{4}$

(d) all of the above are *technically* correct

The Inverse Tangent Function

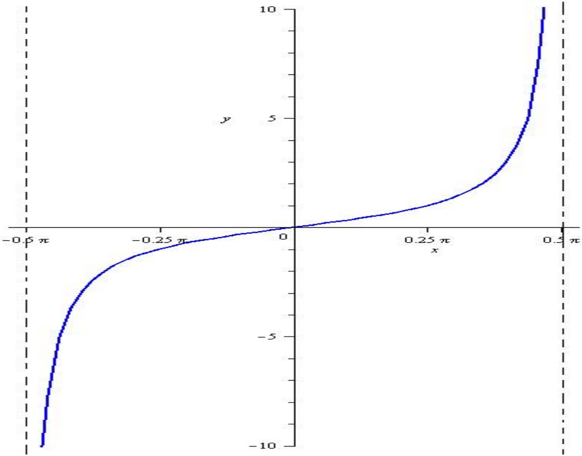


Figure: To define an inverse tangent function, we start by restricting the domain of $\tan(x)$ to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. **(Note the end points are NOT included!)**

The Inverse Tangent Function (a.k.a. arctangent function)

Definition: For all real numbers x, the inverse tangent of x is denoted by

$$\tan^{-1}(x) \quad \text{or by} \quad \arctan(x)$$

quadrants I and IV

and is defined by the relationship

$$y = \tan^{-1}(x) \quad \Longleftrightarrow \quad x = \tan(y) \quad \text{where} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

The Domain of the Inverse Tangent is $-\infty < x < \infty$.

The Range of the Inverse Cosine is $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (Note the strict inequalities.).

The Graph of the Arctangent

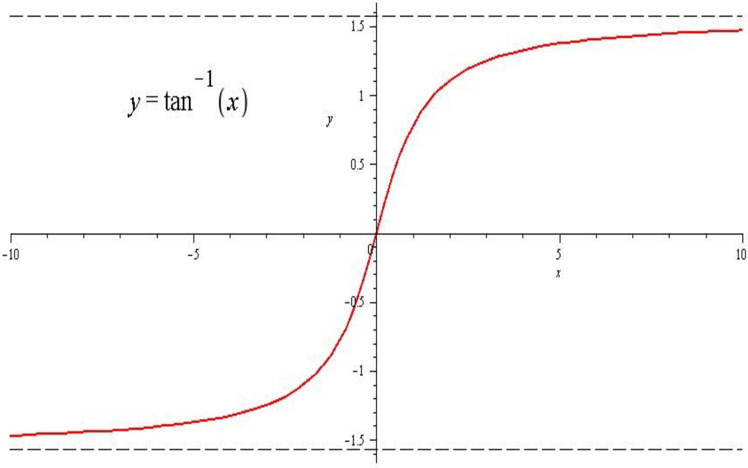


Figure: The domain is all real numbers and the range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$. The graph has two horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.

Conceptual Definition

We can think of the inverse tangent function in the following way:

$\tan^{-1}(x)$ **is the *angle* between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x.**

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is the angle between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is $\frac{1}{\sqrt{3}}$. $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$.

And $\frac{-\pi}{2} < \frac{\pi}{6} < \frac{\pi}{2}$. So

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Question

Both of the following are true statements:

$$\tan\left(\frac{5\pi}{4}\right) = 1 \quad \text{and} \quad \tan\left(\frac{\pi}{4}\right) = 1$$

The value of $\tan^{-1}(1)$ is

(a) $\frac{\pi}{4}$ and $\frac{5\pi}{4}$

(b) just $\frac{5\pi}{4}$

(c) just $\frac{\pi}{4}$

(d) all of the above are *technically* correct

Recap: Inverse Sine, Cosine, and Tangent

Function	Domain	Range
$\sin^{-1}(x)$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Three Other Inverse Trigonometric Functions

There is disagreement about how to define the ranges of the inverse cotangent, cosecant, and secant functions!

The inverse Cotangent function is **typically** defined for all real numbers x by

$$y = \cot^{-1}(x) \quad \Longleftrightarrow \quad x = \cot(y) \quad \text{for} \quad 0 < y < \pi.$$

There is less consensus regarding the inverse secant and cosecant functions.

A Work-Around

Use the fact that $\sec\theta = \frac{1}{\cos\theta}$ to find an expression for $\sec^{-1}(x)$.

Let $\theta = \sec^{-1}(x)$, then $\sec\theta = x$. If $x \neq 0$

then $\frac{1}{\sec\theta} = \frac{1}{x} \Rightarrow \cos\theta = \frac{1}{x}$

$\cos\theta = \frac{1}{x}$ given $\theta = \cos^{-1}\left(\frac{1}{x}\right)$

provided $0 \le \theta \le \pi$ and $x \neq 0$.

Use your result to compute (*an acceptable value for*) $\sec^{-1}(\sqrt{2})$

$$\sec^{-1}(\sqrt{2}) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

A Work-Around

We can use the following compromise to compute inverse cotangent, secant, and cosecant values[2]:

For those values of x for which each side of the equation is defined

$$\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right),$$

$$\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right),$$

$$\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right).$$

[2] **Keep in mind that there is disagreement about the ranges!**

Function/Inverse Function Relationship

For every x in the interval $[-1, 1]$

$$\sin\left(\sin^{-1}(x)\right) = x$$

For every x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\sin^{-1}\left(\sin(x)\right) = x$$

Remark 1: If $x > 1$ or $x < -1$, the expression $\sin^{-1}(x)$ is not defined.

Remark 2: If $x > \frac{\pi}{2}$ or $x < -\frac{\pi}{2}$, the expression $\sin^{-1}\left(\sin(x)\right)$ IS defined, but IS NOT equal to x.

Function/Inverse Function Relationship

For every x in the interval $[-1, 1]$

$$\cos\left(\cos^{-1}(x)\right) = x$$

For every x in the interval $[0, \pi]$

$$\cos^{-1}\left(\cos(x)\right) = x$$

Remark 1: If $x > 1$ or $x < -1$, the expression $\cos^{-1}(x)$ is not defined.

Remark 2: If $x > \pi$ or $x < 0$, the expression $\cos^{-1}\left(\cos(x)\right)$ IS defined, but IS NOT equal to x.

Function/Inverse Function Relationship

For all real numbers x

$$\tan\left(\tan^{-1}(x)\right) = x$$

For every x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\tan^{-1}\left(\tan(x)\right) = x$$

Remark 1: The expression $\tan^{-1}(x)$ is always well defined.

Remark 2: If $x > \frac{\pi}{2}$ or $x < -\frac{\pi}{2}$, the expression $\tan^{-1}\left(\tan(x)\right)$ MAY BE defined, but IS NOT equal to x.

Example

Evaluate each expression if possible. If it is not defined, give a reason.

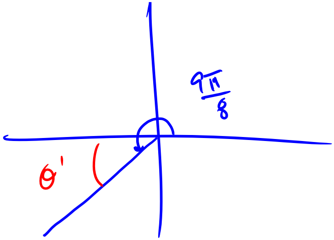
(a) $\sin\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$ $\quad -1 \le \frac{1}{2} \le 1$

$= \frac{1}{2}$

Example

(b) $\cos^{-1}\left[\cos\left(\frac{9\pi}{8}\right)\right]$

$= \frac{7\pi}{8}$



reference angle

$\theta' = \frac{\pi}{8}$

$\cos\left(\frac{7\pi}{8}\right) = \cos\left(\frac{9\pi}{8}\right)$

↑ quad II