March 18 Math 2306 sec. 53 Spring 2019

Section 11: Linear Mechanical Equations

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and (2)} \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$ Suppose $\gamma \neq \omega$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ will this form work?

Yes it will!

The displacement will look like

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + A\cos(\gamma t) + B\sin(\gamma t)$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$ Suppose $\gamma = \omega$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$$x_p = A\cos(\gamma t) + B\sin(\gamma t) = A\cos(\omega t) + B\sin(\omega t)$$

Will this work? No, it duplicates x_c.

$$x_p = At\cos(\omega t) + Bt\sin(\omega t).$$

Then

$$x_p = c_1\cos(\omega t) + c_2\sin(\omega t) + At\cos(\omega t) + Bt\sin(\omega t)$$

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2}\left(\sin(\gamma t) - \frac{\gamma}{\omega}\sin(\omega t)\right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

$$\text{Case (2):} \quad x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t\cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

▸ Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.

Section 12: LRC Series Circuits

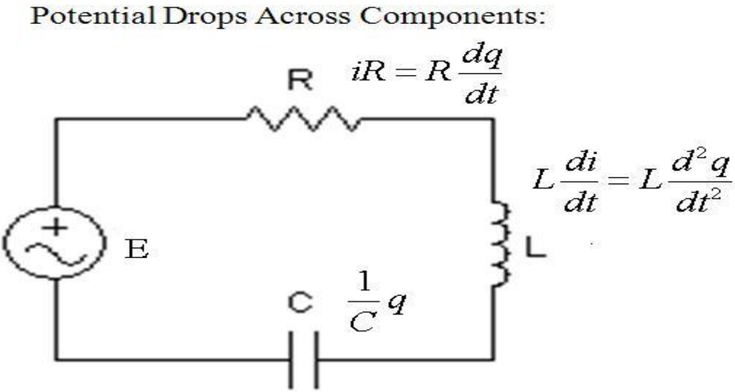

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

This is a second order, linear, constant coefficient nonhomogeneous (if $E \neq 0$) equation.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

$$\begin{array}{lrcl}
\textbf{overdamped} \text{ if} & R^2 - 4L/C & > & 0, \leftarrow \text{2 real roots}\\
\textbf{critically damped} \text{ if} & R^2 - 4L/C & = & 0, \leftarrow \text{1 repeated root}\\
\textbf{underdamped} \text{ if} & R^2 - 4L/C & < & 0.
\end{array}$$

↑ complex conjugate

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system.

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

We're asked for steady state current. This is $\frac{dq_p}{dt}$, i.e. i_p.

The ODE is $Lq'' + Rq' + \frac{1}{C}q = E$. where

$L = \frac{1}{2}$, $R = 10$, $C = 4 \cdot 10^{-3}$ so $\frac{1}{C} = \frac{1}{4 \cdot 10^{-3}} = \frac{10^3}{4} = 250$

We have

$$\frac{1}{2}q'' + 10q' + 250q = 5\cos(10t)$$

In standard form

$$q'' + 20q' + 500q = 10\cos(10t)$$

Let's find q_c: $\quad r^2 + 20r + 500 = 0$

$$r^2 + 20r + 100 - 100 + 500 = 0$$

$$(r+10)^2 + 400 = 0$$

$$(r+10)^2 = -400$$

$$r+10 = \pm 20i$$

$$r = -10 \pm 20i$$

q_c will be

$$q_c = c_1 e^{-10t}\cos(20t) + c_2 e^{-10t}\sin(20t)$$

Now find g_p given $g(t) = 10\cos(10t)$

Using undetermined coefficients,

$$g_p = A\cos(10t) + B\sin(10t)$$

This is correct

Substitute

$$g_p' = -10A\sin(10t) + 10B\cos(10t)$$

$$g_p'' = -100A\cos(10t) - 100B\sin(10t)$$

$$g_p'' + 20g_p' + 500g_p = 10\cos(10t)$$

$$-100A\cos(10t) - 100B\sin(10t) + 20\left(-10A\sin(10t) + 10B\cos(10t)\right)$$

$$+500\left(A\cos(10t) + B\sin(10t)\right) = 10\cos(10t)$$

Collect $\cos(10t)$ and $\sin(10t)$

$$\cos(10t)\left(400A + 200B\right) + \sin(10t)\left(-200A + 400B\right) =$$

$$10\cos(10t) + 0\cdot\sin(10t)$$

Matching gives

$$400A + 200B = 10$$

$$-200A + 400B = 0$$

$$400A + 200B = 10$$
$$-400A + 800B = 0$$
$$1000B = 10 \Rightarrow B = \frac{10}{1000} = \frac{1}{100}$$

$$200A = 400B \Rightarrow A = 2B = 2\left(\frac{1}{100}\right) = \frac{2}{100}$$

So

$$q_p = 0.02\cos(10t) + 0.01\sin(10t)$$

The steady state current

$$\frac{dq_p}{dt} = -10(0.02)\sin(10t) + 10(0.01)\cos(10t)$$

$$\Rightarrow \quad i_p = 0.1\cos(10t) - 0.2\sin(10t)$$