March 1 Math 2306 sec. 54 Spring 2019

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

Recall that we assumed $y_p = Ax + B$ because this is the basic format for $g(x) = 8x + 1$. Upon substitution, we found that $A = 2$ and $B = \frac{9}{4}$ giving

$$y_p = 2x + \frac{9}{4}.$$

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

Here we assumed that $y_p = Ae^{-3x}$ because this is the basic form that $g(x) = 6e^{-3x}$ has. We substituted into the ODE and came up with $A = \frac{6}{25}$ giving

$$y_p = \frac{6}{25}e^{-3x}.$$

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

Suppose we consider $g(x) = 16x^2$ as a monomial, we may assume that $y_p = Ax^2$. Substitute

$y_p' = 2Ax$, $y_p'' = 2A$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2 + 0x + 0$$

Matching gives $4A = 16$, and $-8A = 0$, $2A = 0$

This requires $A = 4$ and $A = 0$ which is impossible!

We need to consider $g(x) = 16x^2$ as a 2^{nd} degree polynomial. Let $y_p = Ax^2 + Bx + C$

$$y_p' = 2Ax + B, \quad y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$4Ax^2 + (-8A + B)x + (2A - 4B + 4C) = 16x^2 + 0x + 0$$

Matching gives

$$\begin{aligned} 4A &= 16 \\ -8A + 4B &= 0 \\ 2A - 4B + 4C &= 0 \end{aligned}$$

$$4A = 16 \Rightarrow A = 4$$

$$-8A + 4B = 0 \Rightarrow B = \frac{8}{4}A = 2A = 2 \cdot 4 = 8$$

$$2A - 4B + 4C = 0 \Rightarrow C = \frac{1}{4}(-2A + 4B) = \frac{1}{4}(-8 + 32)$$

$$= 6$$

So $y_p = 4x^2 + 8x + 6$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0$!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any nonzero constant)

zero degree polynomial

$y_p = A$

(b) $g(x) = x - 7$ 1st degree polynomial

$y_p = Ax + B$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$ 2nd degree polynomial

$$y_p = Ax^2 + Bx + C$$

(d) $g(x) = 3x^3 - 5$ 3rd degree polynomial

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$ 1st degree polynomial times e^{3x}

$$y_p = (Ax+B)e^{3x} = Axe^{3x} + Be^{3x}$$

(f) $g(x) = \cos(7x)$ Linear combo of sine and Cosine of $7x$

$$y_p = A\cos(7x) + B\sin(7x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$

$$y_p = A\sin(2x) + B\cos(2x) + C\cos(4x) + D\sin(4x)$$

(h) $g(x) = x^2\sin(3x)$ 2^{nd} degree poly times sines and cosines of $3x$

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$

$$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$$

(j) $g(x) = xe^{-x} \sin(\pi x)$

$$y_p = (Ax+B)e^{-x} \sin(\pi x) + (Cx+D)e^{-x} \cos(\pi x)$$