March 1 Math 2306 sec. 60 Spring 2019

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

Recall that we assumed $y_p = Ax + B$ because this is the basic format for $g(x) = 8x + 1$. Upon substitution, we found that $A = 2$ and $B = \frac{9}{4}$ giving

$$y_p = 2x + \frac{9}{4}.$$

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

Here we assumed that $y_p = Ae^{-3x}$ because this is the basic form that $g(x) = 6e^{-3x}$ has. We substituted into the ODE and came up with $A = \frac{6}{25}$ giving

$$y_p = \frac{6}{25}e^{-3x}.$$

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

Suppose we consider $g(x) = 16x^2$ as a monomial.

Try setting $y_p = Ax^2$. We substitute

$$y_p' = 2Ax, \quad y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax) + 4Ax^2 = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2 + 0x + 0$$

Matching $\quad 4A = 16, \quad -8A = 0, \quad 2A = 0$

This would require $A=0$ and $A=4$. The choice for y_p won't work. We need an x term and a constant term.

We need to consider $g(x) = 16x^2$ as a quadratic polynomial. Assuming y_p is of this form, put

$$y_p = Ax^2 + Bx + C$$

substitute

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax+B) + 4(Ax^2+Bx+C) = 16x^2$$

$$4A\,x^2 + (-8A+4B)x + (2A-4B+4C) = 16x^2 + 0x + 0$$

Match

$$4A = 16 \Rightarrow A = 4$$

$$-8A + 4B = 0$$

$$2A - 4B + 4C = 0$$

$$4B = 8A \Rightarrow B = 2A = 2(4) = 8$$

$$4C = -2A + 4B = -2(4) + 4(8) = 24$$

$$C = 6$$

The particular solution is

$$y_p = 4x^2 + 8x + 6$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0$!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

Examples of Forms of y_p based on g (Trial Guesses)

nonzero

(a) $g(x) = 1$ (or really any constant)

g is a zero degree polynomial

$y_p = A$

(b) $g(x) = x - 7$

1st degree polynomial

$y_p = Ax + B$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$ 2nd degree polynomial

$$y_p = Ax^2 + Bx + C$$

(d) $g(x) = 3x^3 - 5$ 3rd degree polynomial

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$ 1st degree polynomial times e^{3x}

$$y_p = (Ax + B)e^{3x}$$

(f) $g(x) = \cos(7x)$ Linear combo of Sine and Cosine $7x$

$$y_p = A\cos(7x) + B\sin(7x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$

super position

$$y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$$

(h) $g(x) = x^2\sin(3x)$

2^{nd} degree poly times $\sin(3x)$ and
2^{nd} degree poly times $\cos(3x)$

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$ $\quad e^x$ times $\sin(2x)$ and $\cos(2x)$

$$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$$

(j) $g(x) = xe^{-x} \sin(\pi x)$

$$y_p = (Ax+B)e^{-x} \sin(\pi x) + (Cx+D)e^{-x} \cos(\pi x)$$