March 22 Math 2306 sec. 60 Spring 2019

Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s,t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_\alpha^\beta G(s,t)\,dt$$

- the result is a function of the remaining variable s, and
- the variable s is treated as a constant while integrating with respect to t.

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt + \beta\int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s,t) = e^{-st}$.

Note 2: If we take s to be real-valued, then

$$\lim_{t\to\infty} e^{-st} = 0 \quad \text{if } s > 0, \quad \text{and} \quad \lim_{t\to\infty} e^{-st} = \infty \quad \text{if } s < 0.$$

Find the Laplace transform of $f(t) = 1$

By definition $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$, so

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\,dt = \int_0^\infty e^{-st}\,dt$$

we'll consider $s=0$ and $s \neq 0$ cases.

If $s=0$, the integral is $\int_0^\infty dt = \lim_{b\to\infty} \int_0^b dt = \lim_{b\to\infty} t\Big|_0^b = \infty$

The integral is divergent. Zero is not in the domain.

For $s \neq 0$

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty} \int_0^b e^{-st}\,dt$$

$$= \lim_{b\to\infty} \frac{-1}{s} e^{-st} \Big|_0^b = \lim_{b\to\infty} \frac{-1}{s}\left(e^{-sb} - e^0\right)$$

$$= \frac{-1}{s}(0-1) \quad \text{for } s>0$$

$$= \frac{1}{s}$$

* The integral diverges if $s<0$

The transform $\mathscr{L}\{1\} = \frac{1}{s}$ for $s>0$.

Find the Laplace transform of $f(t) = t$

By definition $\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$

If $s=0$, the integral $\int_0^\infty t\, dt$ is divergent.

For $s \neq 0$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

$$= \frac{-1}{s} e^{-st}\, t \Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st}\, dt$$

Int. by parts

$u = t \qquad du = dt$

$v = \frac{-1}{s} e^{-st} \qquad dv = e^{-st}\, dt$

$$= \frac{-1}{s}(0-0) + \frac{1}{s}\int_0^\infty e^{-st}\,dt \quad \text{for } s>0$$

Integral diverges if $s<0$

$$= \frac{1}{s}\underbrace{\int_0^\infty e^{-st}\,dt}_{\mathcal{L}\{1\}}$$

$$= \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2} \quad \text{for } s>0$$

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^{10} e^{-st} f(t)\,dt + \int_{10}^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st}(2t)\,dt$$

$$= 2\int_0^{10} e^{-st}\, t\,dt$$

For $s=0$, we have $\int_0^{10} 2t\,dt = t^2\Big|_0^{10} = 100$

For $s \neq 0$ $\mathcal{L}\{f(t)\} = 2\int_0^{10} e^{-st}\, t\, dt$

$$= \frac{-2}{s} e^{-st} t \Big|_0^{10} - \frac{2}{s^2} e^{-st} \Big|_0^{10}$$

$$= \frac{-2}{s}\left(e^{-10s} \cdot 10 - 0\right) - \frac{2}{s^2}\left(e^{-10s} - e^{0}\right)$$

$$= \frac{-20}{s} e^{-10s} - \frac{2}{s^2} e^{-10s} + \frac{2}{s^2}$$

So

$$F(s) = \begin{cases} 100\,, & s = 0 \\ \frac{2}{s^2} - \frac{2}{s^2} e^{-10s} - \frac{20}{s} e^{-10s}\,, & s \neq 0 \end{cases}$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t)+\beta g(t)\}=\alpha F(s)+\beta G(s)$
- $\mathscr{L}\{1\}=\frac{1}{s},\quad s>0$
- $\mathscr{L}\{t^n\}=\frac{n!}{s^{n+1}},\quad s>0$ for $n=1,2,\dots$
- $\mathscr{L}\{e^{at}\}=\frac{1}{s-a},\quad s>a$
- $\mathscr{L}\{\cos kt\}=\frac{s}{s^2+k^2},\quad s>0$
- $\mathscr{L}\{\sin kt\}=\frac{k}{s^2+k^2},\quad s>0$