March 25 Math 2306 sec. 54 Spring 2019

Section 13: The Laplace Transform

Definition: Let $f(t)$ be defined on $[0,\infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t)+\beta g(t)\}=\alpha F(s)+\beta G(s)$
- $\mathscr{L}\{1\}=\frac{1}{s}, \quad s>0$
- $\mathscr{L}\{t^n\}=\frac{n!}{s^{n+1}}, \quad s>0$ for $n=1,2,\dots$
- $\mathscr{L}\{e^{at}\}=\frac{1}{s-a}, \quad s>a$
- $\mathscr{L}\{\cos kt\}=\frac{s}{s^2+k^2}, \quad s>0$
- $\mathscr{L}\{\sin kt\}=\frac{k}{s^2+k^2}, \quad s>0$

Examples: Evaluate the Laplace transform of

(a) $f(t) = \cos(\pi t)$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2}$$

Examples: Evaluate

$\mathcal{L}\{1\} = \frac{1}{s}$, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$$\mathcal{L}\{2t^4 - e^{-5t} + 3\} = 2\mathcal{L}\{t^4\} - \mathcal{L}\{e^{-5t}\} + 3\mathcal{L}\{1\}$$

$$2\left(\frac{4!}{s^{4+1}}\right) - \frac{1}{s-(-5)} + 3\left(\frac{1}{s}\right)$$

$$= \frac{48}{s^5} - \frac{1}{s+5} + \frac{3}{s}$$

Examples: Evaluate

(c) $f(t) = (2-t)^2$

Distribute first

$f(t) = 4 - 4t + t^2$

$\mathcal{L}\{(2-t)^2\} = \mathcal{L}\{4 - 4t + t^2\}$

$= 4\mathcal{L}\{1\} - 4\mathcal{L}\{t\} + \mathcal{L}\{t^2\}$

$= 4\left(\frac{1}{s}\right) - 4\left(\frac{1!}{s^{1+1}}\right) + \frac{2!}{s^{2+1}} = \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

f grows no faster than an exponential as $t \to \infty$.

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0,\infty)$ and of exponential order c for some $c>0$, then f has a Laplace transform for $s>c$.

An example of a function that is NOT of exponential order for any c is $f(t)=e^{t^2}$. Note that

$$f(t)=e^{t^2}=\left(e^t\right)^t \quad \Longrightarrow \quad |f(t)|>e^{ct} \quad \text{whenever} \quad t>c.$$

This is a function that doesn't have a Laplace transform. We won't be dealing with this type of function here.

Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \dots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets $\{\}$ **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

We'll use $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\}$

i.e. $\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = t^6$

Note that

$$\frac{1}{s^7} = \frac{1}{6!}\frac{6!}{s^7}$$

so

$$\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!}\,\frac{6!}{s^7}\right\} = \frac{1}{6!}\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{1}{6!}t^6$$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+9} + \dfrac{1}{s^2+9}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{3}\,\frac{3}{s^2+3^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \cos(3t) + \frac{1}{3}\sin(3t)$$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

We'll do a partial fraction decomp on $\dfrac{s-8}{s^2-2s}$

$$\frac{s-8}{s^2-2s} = \frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$$s-8 = A(s-2) + Bs$$

Set $s=0$ $\quad -8 = -2A \implies A = 4$

$s=2$ $\quad -6 = 2B \implies B = -3$

Then

$$\mathcal{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4(1) - 3\left(e^{2t}\right)$$

$$= 4 - 3e^{2t}$$

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\} = \frac{2}{s^3}$?

By properties of exponents

$$e^{-st}\cdot e^{t} = e^{-st+t}$$

$$= e^{-(s-1)t}$$

By definition $\quad \mathscr{L}\left\{e^t t^2\right\} = \int_0^\infty e^{-st}e^t t^2\,dt$

$$= \int_0^\infty e^{-(s-1)t}\,t^2\,dt$$

$$= \int_0^\infty e^{-wt}\,t^2\,dt \quad \text{for } w=s-1$$

Observe that this is simply the Laplace transform of $f(t) = t^2$ evaluated at $s-1$. Letting $F(s) = \mathscr{L}\left\{t^2\right\}$, we have

$$F(s-1) = \frac{2}{(s-1)^3}.$$

Theorem (translation in s)

Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$