March 27 MATH 1112 sec. 54 Spring 2019

Section 7.1: Fundamental Identities: Pythagorean, Sum, and Difference

We have three new identities added to our list. The Pythagorean IDs are

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$

and

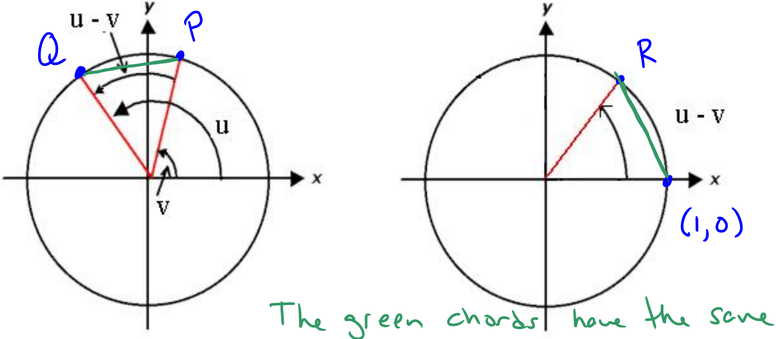
$$1 + \cot^2 x = \csc^2 x$$

Next, we'd like a way to evaluate trigonometric values of sums and differences of angles. For example, what is

$$\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)?$$

Sum and Difference Identities

Given two *angles* u and v, we wish to find a formula for $\cos(u - v)$.



The green chords have the same length

Figure: We construct the angle $u - v$ in a unit circle in two ways and equate the resulting chord lengths.

Derive the formula for $\cos(u - v)$

$P = (\cos v, \sin v)$ $\quad$ $Q = (\cos u, \sin u)$

$R = (\cos(u-v), \sin(u-v))$ $\quad$ Let $S = (1,0)$

Note $|PQ| = |RS|$ (chord lengths)

we'll use $|PQ|^2 = |RS|^2$

$$|PQ|^2 = (\cos u - \cos v)^2 + (\sin u - \sin v)^2$$

$$= \cos^2 u - 2\cos u \cos v + \cos^2 v + \sin^2 u - 2\sin u \sin v + \sin^2 v$$

$$= (\cos^2 u + \sin^2 u) + (\cos^2 v + \sin^2 v) - 2(\cos u \cos v + \sin u \sin v)$$

Derive the formula for $\cos(u - v)$

$$|PQ|^2 = 2 - 2(\cos u \cos v + \sin u \sin v)$$

Next,

$$|RS|^2 = (\cos(u-v) - 1)^2 + (\sin(u-v) - 0)^2$$

$$= \cos^2(u-v) - 2\cos(u-v) + 1 + \sin^2(u-v)$$

$$= (\cos^2(u-v) + \sin^2(u-v)) + 1 - 2\cos(u-v)$$

$$= 2 - 2\cos(u-v)$$

Now set $|PQ|^2 = |RS|^2$

Derive the formula for $\cos(u - v)$

$$2 - 2(\cos u \cos v + \sin u \sin v) = 2 - 2\cos(u-v)$$

$$2\cos(u-v) = 2(\cos u \cos v + \sin u \sin v)$$

$$\boxed{\cos(u-v) = \cos u \cos v + \sin u \sin v}$$

Question

We know that $\cos(u-v) = \cos u \cos v + \sin u \sin v$. Use the fact that

$$\cos(u+v) = \cos(u-(-v)) = \cos u \cos(-v) + \sin u \sin(-v)$$

and the even/odd symmetry of the sine and cosine to deduce the sum of angles formula for the cosine. The result is

$$\cos(u+v) =$$

(a) $\cos u \sin v + \sin u \cos v$

(b) $\cos u + \cos v$

$\cos(-v) = \cos v$

(c) $\cos u \cos v - \sin u \sin v$

$\sin(-v) = -\sin v$

(d) $\cos u + \sin v$

Sum and Difference Identities

Cosine Identities:

$$\text{(sum)} \quad \cos(u+v) = \cos u \cos v - \sin u \sin v,$$

$$\text{(diff)} \quad \cos(u-v) = \cos u \cos v + \sin u \sin v$$

Sine Identities:

$$\text{(sum)} \quad \sin(u+v) = \sin u \cos v + \sin v \cos u$$

$$\text{(diff)} \quad \sin(u-v) = \sin u \cos v - \sin v \cos u$$

Tangent Identities:

$$\text{(sum)} \quad \tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v},$$

$$\text{(diff)} \quad \tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Determine the Exact Value of Each Expression

$\cos(15^\circ)$

We can use $15^\circ = 45^\circ - 30^\circ$

$$= \cos(45^\circ - 30^\circ) = \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Question

The expression $\sin(27^\circ)\cos(10^\circ) - \sin(10^\circ)\cos(27^\circ)$ is equivalent to

(a) $\sin(37^\circ)$

(b) $\cos(37^\circ)$

(c) $\sin(17^\circ)$

(d) $\cos(17^\circ)$

Evaluate each expression using the given information.

Given: $\tan\alpha = -2, \quad \frac{\pi}{2} < \alpha < \pi$ quad II

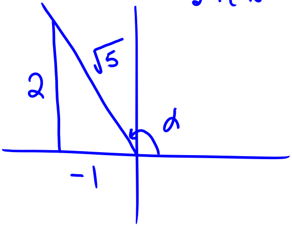
and $\sec\beta = \frac{4}{3}, \quad \frac{3\pi}{2} < \beta < 2\pi$ quad IV

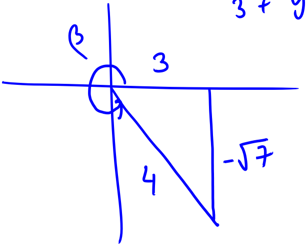
$\tan(\alpha+\beta)$

we'll use representative diagrams

$2^2+(-1)^2=5$

$3^2+y^2=4^2$





$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$\tan\alpha = -2$

$\tan\beta = \frac{-\sqrt{7}}{3}$

$$= \frac{-2 + \left(\frac{-\sqrt{7}}{3}\right)}{1 - (-2)\left(\frac{-\sqrt{7}}{3}\right)} \left(\frac{3}{3}\right)$$

Clear the fractions

$$= \frac{-6 - \sqrt{7}}{3 - 2\sqrt{7}}$$

Section 7.2: Double & Half Angle IDs

Use $\sin(u+v) = \sin u \cos v + \sin v \cos u$ to obtain a formula for

$\sin(2u)$ $= \sin(u+u)$

$= \sin u \cos u + \sin u \cos u$

$= 2 \sin u \cos u$

Double Angle Formulas for the Cosine

Use $\cos(u+v) = \cos u \cos v - \sin u \sin v$ and $\cos^2 u + \sin^2 u = 1$ to find three formulas for

$\cos(2u) = \cos(u+u) = \cos u \cos u - \sin u \sin u$

$$\cos(2u) = \cos^2 u - \sin^2 u$$

Using $\sin^2 u = 1 - \cos^2 u$

$$\cos(2u) = \cos^2 u - (1 - \cos^2 u)$$

$$= \cos^2 u - 1 + \cos^2 u$$

$$= 2\cos^2 u - 1$$

Using $\cos^2 u = 1 - \sin^2 u$

$$\cos(2u) = (1 - \sin^2 u) - \sin^2 u$$

$$= 1 - \sin^2 u - \sin^2 u$$

$$= 1 - 2\sin^2 u$$

Question: Double Angle Formulas for the Tangent

From the sum formula $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$, it follows that

$\tan(2u) =$

(a) $\dfrac{2\tan u}{1 - 2\tan u}$

(b) $\dfrac{2\tan u}{1 - \tan^2 u}$

(c) $\dfrac{\tan^2 u}{1 - 2\tan u}$

(d) $\dfrac{\tan^2 u}{1 - \tan^2 u}$

Double Angle Formulas

$$\sin(2u) = 2\sin u \cos u$$

$$\begin{aligned}\cos(2u) &= \cos^2 u - \sin^2 u\\ &= 2\cos^2 u - 1\\ &= 1 - 2\sin^2 u\end{aligned}$$

$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$