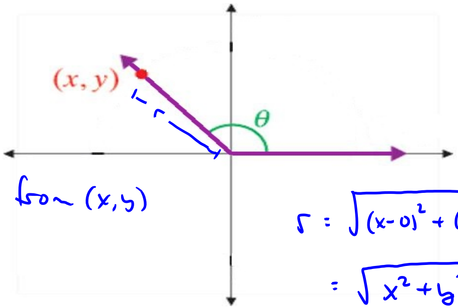
March 4 MATH 1112 sec. 54 Spring 2019

Section 6.3 Trigonometric Functions of any Angle

We wish to extend the definitions of the six trigonometric functions to angles that are not necessarily acute. To start, consider an angle in standard position, and choose a point (x, y) on the terminal side.



r is distance from (x,y) to $(0,0)$

$$r = \sqrt{(x-0)^2 + (y-0)^2}$$
$$= \sqrt{x^2 + y^2}$$

Trigonometric Function of any Angle

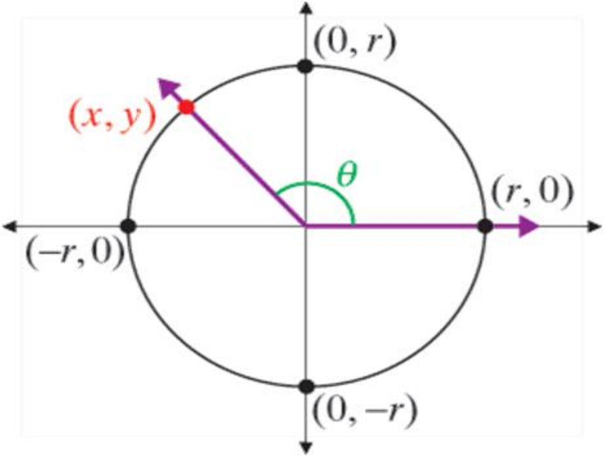


Figure: An angle in standard position determined by a point (x, y). Any such point lives on a circle in the plane centered at the origin having radius $r = \sqrt{x^2 + y^2}$

Trigonometric Function of any Angle

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x} \quad (\text{for } x \neq 0)$$

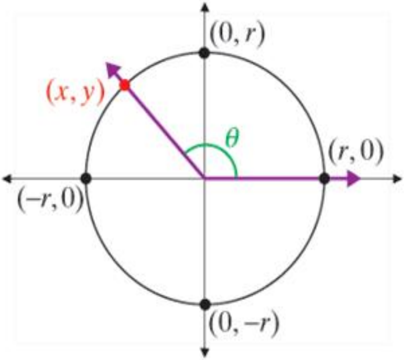


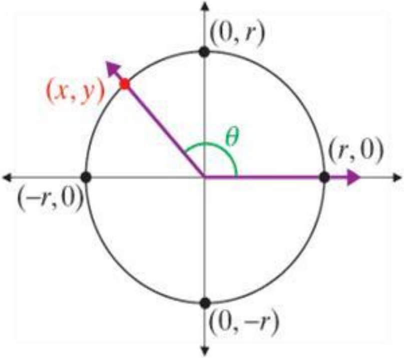
Figure: The definitions for the sine, cosine and tangent of any angle θ are given in terms of x, y, and r.

Trigonometric Function of any Angle

$$\csc\theta = \frac{r}{y} \quad (\text{for } y \neq 0)$$

$$\sec\theta = \frac{r}{x} \quad (\text{for } x \neq 0)$$

$$\cot\theta = \frac{x}{y} \quad (\text{for } y \neq 0)$$



Example

The terminal side of an angle θ in standard position passes through the point $(1, -4)$. Determine the sine and cosine of the angle.

$$\sin\theta = \frac{y}{r} \quad \text{and} \quad \cos\theta = \frac{x}{r}$$

$$x = 1, \quad \text{and} \quad y = -4 \quad \text{so} \quad r = \sqrt{1^2 + (-4)^2} = \sqrt{17}$$

Hence

$$\sin\theta = \frac{-4}{\sqrt{17}} \quad \text{and} \quad \cos\theta = \frac{1}{\sqrt{17}}$$

Question

When put in standard position, the terminal side of the angle θ passes through the point $(-2, 3)$. The sine value of θ is

(a) $\sin\theta = -\dfrac{3}{2}$

$y = 3$, $r = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$

(b) $\sin\theta = \dfrac{3}{\sqrt{13}}$

(c) $\sin\theta = \dfrac{3}{5}$

(d) $\sin\theta = \dfrac{3}{\sqrt{5}}$

Reciprocal Identities

We have the first in a long list of **trigonometric identities**:

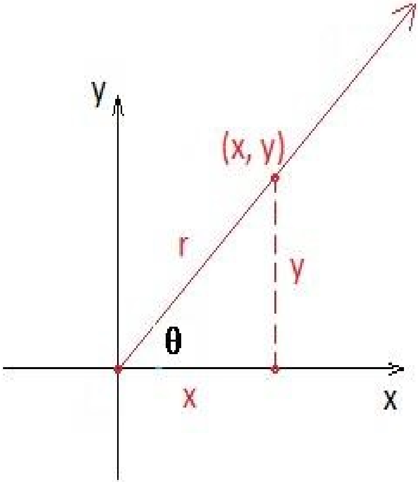
Reciprocal Identities: For any given θ for which both sides are defined

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \& \quad \cot\theta = \frac{1}{\tan\theta}.$$

Equivalently

$$\sin\theta = \frac{1}{\csc\theta}, \quad \cos\theta = \frac{1}{\sec\theta}, \quad \& \quad \tan\theta = \frac{1}{\cot\theta}.$$

Comparison to Acute Angle Definitions



opp $= y$ adj $= x$ and hyp $= r$

OLD Def: $\sin\theta = \frac{\text{opp}}{\text{hyp}}$

New def: $\sin\theta = \frac{y}{r}$

$\Big\}$ equal

Similarly

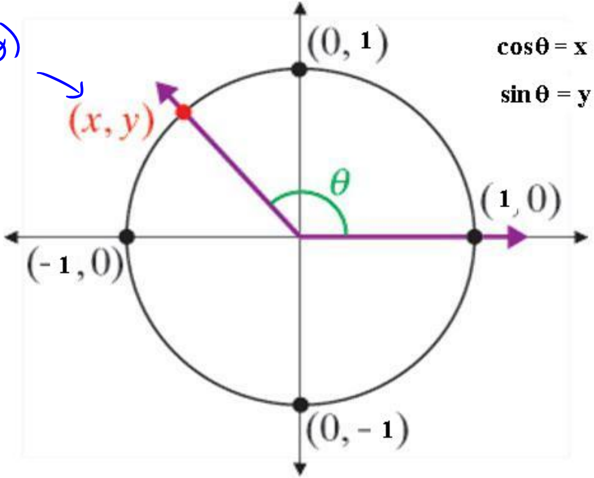
$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$

Figure: Note that the acute angle definitions still hold.

Trigonometric Function of any Angle (Unit Circle Case)

point is $(\cos\theta, \sin\theta)$



$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

Figure: A point on the unit circle, $r = 1$, has coordinates $(x, y) = (\cos\theta, \sin\theta)$.

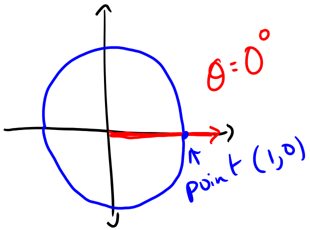
Quadrantal Angles

The angles 0° and 90° both have the property that when put in standard position, the terminal side is concurrent with one of the coordinate axes. Such angles are called **quadrantal angles.** Some other quadrantal angles include

$$180^\circ, \quad 270^\circ, \quad -90^\circ, \quad \text{and} \quad 360^\circ$$

Trigonometric Values of Quadrantal Angles

Determine the six trigonometric values of 0° as possible.



Terminal side sits on initial side

$x = 1,\ y = 0,\ r = \sqrt{1^2 + 0^2} = 1$

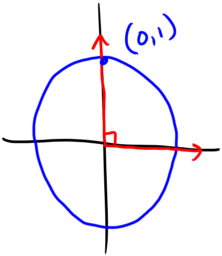
$\sin 0^\circ = \frac{y}{r} = \frac{0}{1} = 0$ $\qquad \csc 0^\circ = \frac{r}{y}$ undefined

$\cos 0^\circ = \frac{x}{r} = \frac{1}{1} = 1$ $\qquad \sec 0^\circ = \frac{r}{x} = \frac{1}{1} = 1$

$\tan 0^\circ = \frac{y}{x} = \frac{0}{1} = 0$ $\qquad \cot 0^\circ = \frac{x}{y}$ undefined

Trigonometric Values of Quadrantal Angles

Determine the six trigonometric values of 90° as possible.



$x = 0$, $y = 1$, $r = 1$

$\sin 90^\circ = \frac{y}{r} = \frac{1}{1} = 1$ $\qquad$ $\csc 90^\circ = \frac{r}{y} = \frac{1}{1} = 1$

$\cos 90^\circ = \frac{x}{r} = \frac{0}{1} = 0$ $\qquad$ $\sec 90^\circ = \frac{r}{x}$ undefined

$\tan 90^\circ = \frac{y}{x}$ undefined $\qquad$ $\cot 90^\circ = \frac{x}{y} = \frac{0}{1} = 0$

A Useful Table of Trigonometric Values

θ°	0°	30°	45°	60°	90°
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef.

Quadrants & Signs

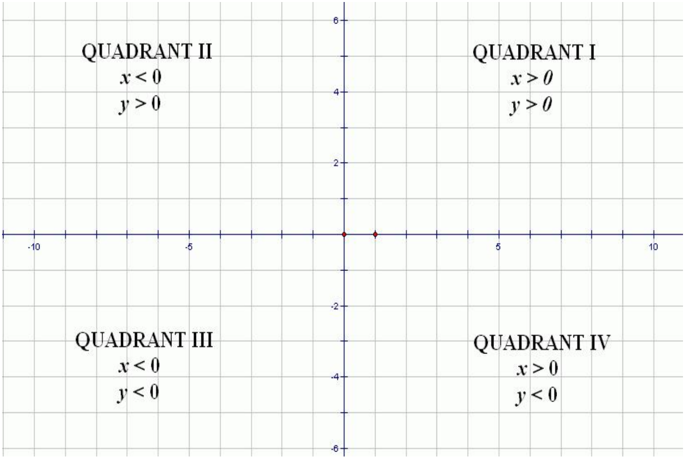


Figure: The trigonometric values for a general angle may be positive, negative, zero, or undefined. The signs are determined by the signs of the x and y values. **Note that $r > 0$ by definition.**

Quadrants & Signs of Trig Values