March 4 Math 2306 sec. 53 Spring 2019

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

By superposition, we know that y_p can be written as

$$y_p = y_{p_1} + y_{p_2}$$

Where y_{p_1} solves $y'' - 4y' + 4y = 6e^{-3x}$ $\leftarrow g_1(x)$ and

y_{p_2} solves $y'' - 4y' + 4y = 16x^2$ $\uparrow g_2(x)$

Based on $g_1(x)$, $y_{p_1} = Ae^{-3x}$

Based on $g_2(x)$, $y_{p_2} = Bx^2 + Cx + D$

The correct y_p for $y'' - 4y' + 4y = 6e^{-3x} + 16x^2$

is

$$y_p = Ae^{-3x} + Bx^2 + Cx + D$$

A Glitch!

$$y'' - y' = 3e^x$$

Based on $g(x) = 3e^x$, we set $y_p = Ae^x$. Substitute

$y_p' = Ae^x$, $y_p'' = Ae^x$

$$y_p'' - y_p' = 3e^x$$

$$Ae^x - Ae^x = 3e^x$$

$$0Ae^x = 3e^x$$

$$0A = 3$$

Can't be solved, no such A exists

If we consider y_c, we'll see why this failed. y_c solves

$$y'' - y' = 0$$

Characteristic equation $\quad m^2 - m = 0$

$$m(m-1) = 0 \implies m_1 = 0 \text{ and } m_2 = 1$$

The complementary solution is $\quad y = c_1 + c_2 e^x$

The choice Ae^x solves the homogeneous equation leaving no way to match with the $3e^x$ on the right.

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.

Case II Examples

Solve the ODE

$$y'' - 2y' + y = -4e^x$$

We're looking for $y = y_c + y_p$. First find y_c.

y_c solves $\quad y'' - 2y' + y = 0$

Characteristic eqn. $\quad m^2 - 2m + 1 = 0 \implies (m-1)^2 = 0$

$m = 1$ repeated

$$y_1 = e^x \quad , \quad y_2 = x e^x$$

$$y_c = c_1 e^x + c_2 x e^x$$

Now find y_p given $g(x) = -4e^x$. At first pass

Set

1st try $y_p = Ae^x$ duplicates c_1e^x, this won't work

2nd try $y_p = Axe^x$ duplicates c_2xe^x, this won't work

3rd try $y_p = Ax^2e^x$, this is the correct form.

Substitute

$$y_p' = Ax^2e^x + 2Axe^x$$

$$y_p'' = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$$

$$= Ax^2e^x + 4Axe^x + 2Ae^x$$

$$y_p'' - 2y_p' + y_p = -4e^x$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2\left(Ax^2e^x + 2Axe^x\right) + Ax^2e^x = -4e^x$$

Collect like terms $\left(x^2e^x,\ xe^x,\ \text{and}\ e^x\right)$

$$x^2e^x\left(\underbrace{A - 2A + A}_{0}\right) + xe^x\left(\underbrace{4A - 4A}_{0}\right) + e^x(2A) = -4e^x$$

$$2Ae^x = -4e^x$$

$$2A = -4$$

$$A = -2$$

So $y_p = -2x^2 e^x$

The general solution $y_c + y_p$ is

$$y = c_1 e^x + c_2 x e^x - 2x^2 e^x$$

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Look @ y_c : $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0$ $m = 2$ repeated

$y_1 = e^{2x}$, $y_2 = xe^{2x}$ $y_c = c_1 e^{2x} + c_2 x e^{2x}$

Let y_{p_1} solve $y'' - 4y' + 4y = \sin(4x)$

$g(x) = \sin(4x)$ so put

$y_{p_1} = A\sin(4x) + B\cos(4x)$ ✓

Let y_{p_2} solve $y'' - 4y' + 4y = xe^{2x}$

$y_c = C_1 e^{2x} + C_2 x e^{2x}$

$g(x) = xe^{2x}$

$y_{p_2} = (Cx + D)e^{2x} = Cxe^{2x} + De^{2x}$ nope!

2nd try $\quad y_{p_2} = (Cx + D)xe^{2x} = Cx^2e^{2x} + Dxe^{2x}$

3rd $\quad y_{p_3} = (Cx + D)x^2e^{2x} = Cx^3e^{2x} + Dx^2e^{2x}$ ✓

$y_p = y_{p_1} + y_{p_2} = A\sin(4x) + B\cos(4x) + Cx^3e^{2x} + Dx^2e^{2x}$