March 4 Math 2306 sec. 54 Spring 2019

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

By superposition, we can write $y_p = y_{p_1} + y_{p_2}$ where

y_{p_1} solves $\quad y'' - 4y' + 4y = 6e^{-3x}$ and

$\uparrow g_1(x)$

y_{p_2} solves $\quad y'' - 4y' + 4y = 16x^2$

$\uparrow g_2(x)$

From $g_1(x) = 6e^{-3x}$, we set

$$y_{p_1} = Ae^{-3x}$$

From $g_2(x) = 16x^2$, we set

$$y_{p_2} = Bx^2 + Cx + D$$

Then $y_p = Ae^{-3x} + Bx^2 + Cx + D$

A Glitch!

$$y'' - y' = 3e^x$$

Here, $g(x) = 3e^x$, we'd set $y_p = Ae^x$. Substitute

$y_p' = Ae^x$, $y_p'' = Ae^x$

$$y_p'' - y_p' = 3e^x$$

$$Ae^x - Ae^x = 3e^x$$

$$0Ae^x = 3e^x$$

$$0A = 3$$

Can't be solved, there is no solution A.

We need to consider y_c which solves $y''-y'=0$

Characteristic equation $m^2-m=0$

$m(m-1)=0 \Rightarrow m_1=0,\ m_2=1$

$$y_1=e^{0x}=1 \quad \text{and} \quad y_2=e^{1x}=e^x$$

$$y_c=c_1+c_2e^x$$

Our guess $y_p=Ae^x$ solves the homogeneous equation (matches part of y_c). Theres no way to match to the $g(x)=3e^x$.

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.

Case II Examples

Solve the ODE

$$y'' - 2y' + y = -4e^x$$

We need $y = y_c + y_p$. Find y_c first. y_c solves

$y'' - 2y' + y = 0$ characteristic equation

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1$ repeated

$y_1 = e^x$, $y_2 = xe^x$

$y_c = c_1 e^x + c_2 x e^x$

Find y_p: $g(x) = -4e^x$. Guess at the form of y_p

1st try $y_p = Ae^x$ Duplicates c_1e^x, won't work

2nd try $y_p = Axe^x$ Duplicates c_2xe^x, won't work

3rd try $y_p = Ax^2e^x$ this is the correct form

Substitute

$$y_p' = Ax^2e^x + 2Axe^x$$

$$y_p'' = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$$

$$= Ax^2e^x + 4Axe^x + 2Ae^x$$

$$y_p'' - 2y_p' + y_p = -4e^x$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2\left(Ax^2e^x + 2Axe^x\right) + Ax^2e^x = -4e^x$$

Collect like terms $\left(x^2e^x,\ xe^x,\ e^x\right)$

$$x^2e^x\left(\underbrace{A-2A+A}_{=0}\right) + xe^x\left(\underbrace{4A-4A}_{=0}\right) + e^x(2A) = -4e^x$$

$$2Ae^x = -4e^x$$

$$2A = -4$$

$$A = -2$$

So $y_p = -2x^2e^x$

The general solution $y_c + y_p$ is

$$\boxed{y = c_1 e^x + c_2 x e^x - 2x^2 e^x}$$

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

We need y_c first. y_c solves $y'' - 4y' + 4y = 0$

$m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0$ $m = 2$ repeated

$y_1 = e^{2x}$, $y_2 = xe^{2x}$ $y_c = c_1 e^{2x} + c_2 x e^{2x}$

Let $g_1(x) = \sin(4x)$, $g_2(x) = xe^{2x}$

y_{p_1} solves $y_{p_1}'' - 4y_{p_1}' + 4y_{p_1} = \sin(4x)$

$y_p = A\sin(4x) + B\cos(4x)$ ✓ works

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

$$g_2(x) = x e^{2x}$$

1st attempt $\quad y_{p_2} = (Cx + D) e^{2x} = Cx e^{2x} + De^{2x}$ $\quad$ nope!

2nd $\quad y_{p_2} = (Cx + D) x e^{2x} = Cx^2 e^{2x} + Dx e^{2x}$ $\quad$ nope!

3rd $\quad y_{p_2} = (Cx + D) x^2 e^{2x} = Cx^3 e^{2x} + Dx^2 e^{2x}$ $\quad$ ✓

Then $y_p = y_{p_1} + y_{p_2}$

$$y_p = A\sin(4x) + B\cos(4x) + Cx^3 e^{2x} + Dx^2 e^{2x}$$