March 4 Math 2306 sec. 60 Spring 2019

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

By superposition, we can find y_p in the form

$$y_p = y_{p_1} + y_{p_2}$$

where y_{p_1} solves $y'' - 4y' + 4y = 6e^{-3x}$ $\leftarrow g_1(x)$ and

y_{p_2} solves $y'' - 4y' + 4y = 16x^2$ $\leftarrow g_2(x)$

We know that the correct form of y_{p_1} is

$$y_{p_1} = Ae^{-3x}$$

and the correct form for y_{p_2} is

$$y_{p_2} = Bx^2 + Cx + D$$

For $y'' - 4y' + 4y = 6e^{-3x} + 16x^2$, the y_p form

is $$y_p = Ae^{-3x} + Bx^2 + Cx + D$$

A Glitch!

$$y'' - y' = 3e^x$$

Here $g(x) = 3e^x$, so we can try $y_p = Ae^x$. We substitute $y_p' = Ae^x$, $y_p'' = Ae^x$

$$y_p'' - y_p' = 3e^x$$

$$Ae^x - Ae^x = 3e^x$$

$$0e^x = 3e^x$$

It's not possible to choose A so that $0A = 3$.

We need to look @ y_c. y_c solves

$$y_c'' - y_c' = 0$$

Characteristic eqn. $m^2 - m = 0 \Rightarrow m(m-1) = 0$

$m_1 = 0$, $m_2 = 1$, $y_1 = e^{0x} = 1$, $y_2 = e^x$

Our choice $y_p = Ae^x$ solves the homogeneous equation. So we got zero with no chance of matching to the $g(x) = 3e^x$.

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.

Case II Examples

Solve the ODE

$$y'' - 2y' + y = -4e^x$$

we're finding the general solution. Find y_c first.

Characteristic equation: $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1$ repeated

$y_1 = e^x, \quad y_2 = xe^x$

$y_c = c_1 e^x + c_2 x e^x$

Now, we find y_p. $g(x) = -4e^x$

Based on g, the initial guess would by

1st guess $\quad y_p = Ae^x$ Duplicates c_1e^x, won't work

2nd guess $\quad y_p = Axe^x$ Duplicates c_2xe^x, won't work

3rd time $\quad y_p = Ax^2e^x$ no duplication, this is correct

Now we find y_p.

$$y_p = Ax^2e^x$$

$$y_p' = Ax^2e^x + 2Axe^x$$

$$y_p'' = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$$

$$= Ax^2e^x + 4Axe^x + 2Ae^x$$

$$y_p'' - 2y_p' + y_p = -4e^x$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = -4e^x$$

Collect like terms $(x^2e^x,\ xe^x,$ and $e^x)$

$$x^2e^x\underbrace{(A-2A+A)}_{=0} + xe^x\underbrace{(4A-4A)}_{=0} + e^x(2A) = -4e^x$$

$$2Ae^x = -4e^x$$

$$2A = -4$$

$$A = -2$$

So $\quad y_p = -2x^2e^x$

The general solution $y = y_c + y_p$

$$\boxed{y = c_1 e^x + c_2 x e^x - 2x^2 e^x}$$