March 6 Math 2306 sec. 53 Spring 2019

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4 - 7x^2$$

The characteristic equation $m^3 - m^2 + m - 1 = 0$ factors as $(m-1)(m^2+1) = 0$. So the roots are $m_1 = 1$ and $m_{2,3} = \pm i$.

$y_1 = e^x$, $y_2 = e^{0x}\cos(x)$, $y_3 = e^{0x}\sin(x)$ $\alpha = 0$, $\beta = 1$

$y_c = c_1 e^x + c_2 \cos x + c_3 \sin x$

Letting $g_1(x) = \cos x$, Look for y_{p_1}

set $y_{p_1} = A\cos x + B\sin x$ Duplicates part of y_c

$y_{p_1} = (A\cos x + B\sin x)x = Ax\cos x + Bx\sin x$

Correct form

Letting $g_2(x) = x^4 - 7x^2$

$$y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$$ correct form

$$\boxed{\begin{array}{l} y_p = y_{p_1} + y_{p_2} \\ y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G \end{array}}$$

Find the form of the particular soluition

$$y'' - 2y' + 5y = e^x + 7\sin(2x)$$

The characteristic equation is $m^2 - 2m + 5 = 0$ with roots, $m = 1 \pm 2i$.

$y_1 = e^x \cos(2x)$, $y_2 = e^x \sin(2x)$

$y_c = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)$

Letting $g_1(x) = e^x$ $\qquad y_{p_1} = Ae^x$

Letting $g_2(x) = 7\sin(2x)$ $\qquad y_{p_2} = B\sin(2x) + C\cos(2x)$

$$y_p = Ae^{x} + B\sin(2x) + C\cos(2x)$$

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2y'' + xy' - 4y = e^x.$$

The method of undetermined coefficients is not applicable to either of these. We require another approach.

tan x is not the right kind of right hand side

The 2nd equation is not constant coefficient.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

$y_c = c_1 y_1(x) + c_2 y_2(x)$

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We have 2 unknowns u_1 and u_2, but only one equation. We'll introduce a second equation in such a way as to simplify the work.

Differentiate

$$y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}_{\text{set to zero}}$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

Our 2^{nd} equation will be

$$u_1' y_1 + u_2' y_2 = 0$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

$$y_p'' + P(x) y_p' + Q(x) y_p = g(x)$$

$$u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + P(x)\left(u_1 y_1' + u_2 y_2'\right) + Q(x)\left(u_1 y_1 + u_2 y_2\right) = g(x)$$

Collect by u_1, u_2, u_1', and u_2'

$$u_1\left(\underbrace{y_1''+P(x)y_1'+Q(x)y_1}_{=0}\right)+u_2\left(\underbrace{y_2''+P(x)y_2'+Q(x)y_2}_{=0}\right)+u_1'y_1'+u_2'y_2'=g(x)$$

since y_1 and y_2 solve the homogeneous equation

The 2 equations for u_1 and u_2 are

$$u_1'y_1+u_2'y_2=0$$

$$u_1'y_1'+u_2'y_2'=g$$

Will solve with Cramer's rule.

In matrix form, the system is

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

Let
$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = 0 - g y_2 = -g y_2$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = y_1 g - 0 = g y_1$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$ The wronskian of y_1, y_2

By linear independence, we know $W \neq 0$.

Then $u_1' = \dfrac{W_1}{W}$ and $u_2' = \dfrac{W_2}{W}$

$$u_1 = \int \frac{-g(x)\, y_2(x)}{W}\, dx \quad \text{and} \quad u_2 = \int \frac{g(x)\, y_1(x)}{W}\, dx$$

$$y_p = u_1 y_1 + u_2 y_2$$

Example:

Solve the ODE $y'' + y = \tan x$.

Need y_c: $\quad y_c'' + y_c = 0 \qquad m^2 + 1 = 0$

$$m^2 = -1 \Rightarrow m = \pm i$$

$$\alpha = 0, \quad \beta = 1$$

$y_1 = \cos x \qquad y_2 = \sin x$

we need g and W.

The ode is in standard form so $g(x) = \tan x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos x(\cos x) - \sin x(-\sin x)$$

$$= \cos^2 x + \sin^2 x = 1$$

$$u_1 = \int \frac{-g(x)\, y_2(x)}{W}\, dx = \int \frac{-\tan x \sin x}{1}\, dx$$

$$= -\int \tan x \sin x \, dx$$

$$u_2 = \int \frac{g(x)\, y_1(x)}{W}\, dx = \int \frac{\tan x \cos x}{1}\, dx$$

$$= \int \tan x \cos x \, dx$$

We'll finish later.