March 6 Math 2306 sec. 60 Spring 2019

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + \underbrace{x^4 - 7x^2}_{g_2(x)}$$

$\leftarrow g_1(x)$

The characteristic equation $m^3 - m^2 + m - 1 = 0$ factors as $(m-1)(m^2+1) = 0$. So the roots are $m_1 = 1$ and $m_{2,3} = \pm i$.

$\alpha = 0, \beta = 1$

$y_1 = e^x$, $y_2 = e^{0x}\cos(x)$, $y_3 = e^{0x}\sin(x)$

$y_c = c_1 e^x + c_2 \cos x + c_3 \sin x$

We can consider $\cos x + x^4 - 7x^2 = g_1(x) + g_2(x)$

y_{p_1} for $g_1(x) = \cos x$

$y_{p_1} = A\cos x + B\sin x$ Duplicates y_c

$y_{p_1} = (A\cos x + B\sin x)x$

$y_p = Ax\cos x + Bx\sin x$ correct form

y_2 for $g_2(x) = x^4 - 7x^2$ $\quad y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$

$$y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$$

Find the form of the particular soluition

$$y'' - 2y' + 5y = e^x + 7\sin(2x)$$

The characteristic equation is $m^2 - 2m + 5 = 0$ with roots, $m = 1 \pm 2i$.

$y_C = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)$ $\alpha = 1, \ \beta = 2$

Let $g_1(x) = e^x$, $g_2(x) = 7\sin(2x)$

y_{p_1} for $g_1(x)$ $y_{p_1} = Ae^x$

y_{p_2} for $g_2(x) = 7\sin(2x)$ $y_{p_2} = B\cos(2x) + C\sin(2x)$

$$y_p = Ae^{x} + B\cos(2x) + C\sin(2x)$$

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2y'' + xy' - 4y = e^x.$$

The method of undetermined coefficients is not applicable to either of these. We require another approach.

$\tan x$ is not a correct right side type

The 2nd equation is not constant coefficient.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

$y_c = c_1 y_1 + c_2 y_2$

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We have 2 unknowns u_1, u_2 but only one equation, the ODE. We'll introduce a 2nd equation to make the derivation a bit simpler.

We have to plug y_p into the ODE. Differentiate

$$y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}_{\text{set to zero}}$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

Our 2^{nd} equation is going to be

$$u_1' y_1 + u_2' y_2 = 0$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Plug into $\quad y_p'' + P(x) y_p' + Q(x) y_p = g(x)$

$$u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + P(x)\left(u_1 y_1' + u_2 y_2'\right) + Q(x)\left(u_1 y_1 + u_2 y_2\right) = g(x)$$

Collect terms u_1', u_2', u_1 and u_2

$$u_1\underbrace{\left(y_1''+P(x)y_1'+Q(x)y_1\right)}_{=0}+u_2\underbrace{\left(y_2''+P(x)y_2'+Q(x)y_2\right)}_{=0}+u_1'y_1'+u_2'y_2'=g(x)$$

y_1, y_2 solve the homogeneous equation

We have two equations for the u_1, u_2

$$u_1'y_1+u_2'y_2=0$$
$$u_1'y_1'+u_2'y_2'=g$$

We'll solve it using Cramer's rule

Written using a matrix, the system is

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

This is uniquely solvable since y_1 and y_2 are linearly independent.

Set

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = 0 - g y_2 = -g y_2$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = y_1 g - 0 = g y_1$$

Letting $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ the Wronskian of y_1, y_2

Then $u_1' = \frac{W_1}{W}$ and $u_2' = \frac{W_2}{W}$

So

$$u_1 = \int \frac{-g(x)\, y_2(x)}{W}\, dx \quad \text{and} \quad u_2 = \int \frac{g(x)\, y_1(x)}{W}\, dx$$

and $y_p = u_1 y_1 + u_2 y_2$

Example:

Solve the ODE $y'' + y = \tan x$.

We need y_c: $\quad y_c'' + y_c = 0$

Characteristic equation $\quad m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i \qquad \alpha = 0, \ \beta = 1$

$y_1 = e^{0x}\cos x \qquad y_2 = e^{0x}\sin x$

$y_1 = \cos x \qquad y_2 = \sin x$

pick an order for y_1, y_2 here, and keep it.

We need $g(x)$ and W. The ODE is in standard form, so $g(x) = \tan x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos x(\cos x) - \sin x(-\sin x)$$

$$= \cos^2 x + \sin^2 x = 1$$

$$u_1 = \int \frac{-g(x)\, y_2(x)}{W}\, dx = \int \frac{-\tan x \sin x}{1}\, dx$$

$$= \int -\tan x \sin x \, dx$$

$$u_2 = \int \frac{g(x)\, y_1(x)}{W} \, dx = \int \frac{\tan x \cos x}{1} \, dx$$

$$= \int \tan x \cos x \, dx$$

will finish at later date.