March 8 Math 2306 sec. 60 Spring 2019

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

We'll consider cases

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + g_2(x) + \cdots + g_k(x)$$

Using superposition as needed, begin with the assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

For each sub-problem:

Case I: y_{p_i} as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_{p_i} includes a term that duplicates a term in the complementary solution. Multiply y_{p_i} by x^n, where n is the smallest positive integer required to eliminate the duplication.

Solve the IVP

$$y'' - y = 4e^{-x} + 5 \quad y(0) = 0, \quad y'(0) = 0$$

Get y_c: $m^2 - 1 = 0 \Rightarrow (m-1)(m+1) = 0$ $\quad m_1 = 1, \; m_2 = -1$

so $y_c = c_1 e^{x} + c_2 e^{-x}$

y_{p_1}: Solve $y'' - y = 4e^{-x}$ $\quad g_1(x) = 4e^{-x}$

try $y_{p_1} = Ae^{-x}$ Duplicates $c_2 e^{-x}$

$y_{p_1} = Axe^{-x}$ this is the correct form

$y_{p_1}' = -Axe^{-x} + Ae^{-x}$, $\quad y_{p_1}'' = Axe^{-x} - Ae^{-x} - Ae^{-x} = Axe^{-x} - 2Ae^{-x}$

$y_{p_1}'' - y_p = 4e^{-x}$

$$Axe^{-x} - 2Ae^{-x} - Axe^{-x} = 4e^{-x}$$

$$-2Ae^{-x} = 4e^{-x} \Rightarrow A = -2$$

So $y_{p_1} = -2x \cdot e^{-x}$

y_{p_2} : Solve $y'' - y = 5$ $\quad g_2(x) = 5$

Set $y_{p_2} = A$ this is correct

$y_{p_2}' = 0$, $y_{p_2}'' = 0$

$$y_{p_2}'' - y_{p_2} = 5$$

$$0 - A = 5 \Rightarrow A = -5$$

So $y_{p_2} = -5$

$$y_p = y_{p_1} + y_{p_2} = -2xe^{-x} - 5$$

and the general solution to the ODE is

$$y = c_1 e^{x} + c_2 e^{-x} - 2xe^{-x} - 5 .$$

Now we apply the initial conditions

$$y(0) = 0 , \quad y'(0) = 0$$

$$y(x) = c_1 e^{x} + c_2 e^{-x} - 2xe^{-x} - 5$$

$$y'(x) = c_1 e^{x} - c_2 e^{-x} - 2e^{-x} + 2xe^{-x}$$

$$y(0) = c_1 + c_2 - 5 = 0$$

$$y'(0) = c_1 - c_2 - 2 = 0$$

$$\Longrightarrow \quad \begin{aligned} c_1 + c_2 &= 5 \\ c_1 - c_2 &= 2 \end{aligned}$$

Sum $\quad 2c_1 = 7 \qquad c_1 = \frac{7}{2}$

$$c_2 = 5 - c_1 = \frac{10}{2} - \frac{7}{2} = \frac{3}{2}$$

Finally, the solution to the IVP is

$$y = \frac{7}{2}e^{x} + \frac{3}{2}e^{-x} - 2xe^{-x} - 5$$