NEWTON'S METHOD

EVAN BANYASH, CHANDLER BOGOLIN, AND DONNA YOUNG

This paper is dedicated to Dr. Ritter. We love him so.

Abstract. For this paper, we've explored Newton's Method. We have enumerated a few applications and a means by which it is possible to determine how good an initial estimate must be for it to work. Newton's method is a means of iteratively approximating a root of a function. This has many applications in approximating the solutions to equations of all kinds, so long as they can be expressed in the form of a root finding problem. It is a very basic algorithm and has hence been superseded, but it's still worth studying and is still a powerful way of estimating a root, with both pinpoint precision and efficiency, even though it is based off of the simple idea of linear approximation. As you will see throughout this paper, it can be a very reliable method for very accurate approximations. However, it is not a "cure-all" for finding the root of a function. Like many approximating algorithms, it has its fair share of quirks and oddities.

Introduction

We will first explore the general form of Newton's Method, demonstrate its flexibility by using it to approximate an intersection point, and show how it can be applied to an unconventional problem involving performing the operation of division without actually taking a ratio. We will then demonstrate the shortcomings of the method when poor estimates are used, and finally use it to produce some interesting series of iterate values.

1. Root Approximation

It is possible to approximate a root α of a function $f(x)$ by choosing an arbitrary x_0 that can be assumed to be *somewhere close to* α. Then, successive linear approximations can be taken to get a better and better estimate of the actual location of the root.

Theorem 1.1. *The linear approximation of function f at point x_0.*

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$L(x)$ may then be taken at $x = 0$ to find a succeeding x_1, the location of the approximation's root.

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Theorem 1.2. *The derivation of a step of Newton's Law from a linear approximation.*

$$
\begin{aligned}
f(x_0) + f'(x_0)(x_1 - x_0) &= 0 \\
f'(x_0)(x_1 - x_0) &= -f(x_0) \\
x_1 - x_0 &= \frac{-f(x_0)}{f'(x_0)} \\
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)}
\end{aligned} \tag{1.1}
$$

This works for finding any x_{n+1} from any x_n.

Theorem 1.3. *The general recursive form of Newton's law where n is the current iteration.*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

2. Approximating an Intersection of Two Functions

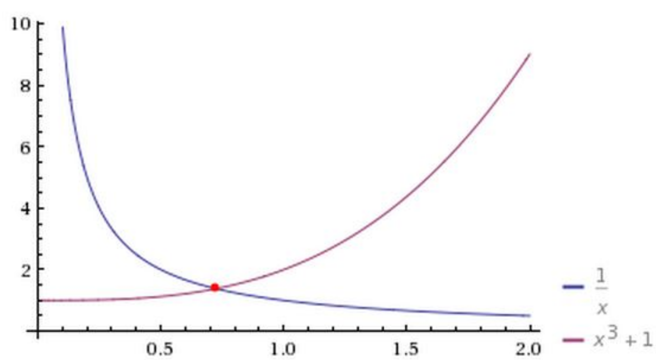

Figure 1. Graph showing one positive intersection of the two graphs.

Newton's Method can also be used to approximate the intersection point of two graphs. The equation $1/x = 1 + x^3$ has one positive solution. If you can set it up as a root finding process, you can find out where they intersect.

$$
\begin{aligned}
\frac{1}{x} &= 1 + x^3 \\
1 &= (1 + x^3)x \\
1 &= x + x^4 \\
0 &= x^4 + x - 1
\end{aligned} \tag{2.1}
$$

Now that we have it set up as a root finding problem, we can apply Newton's Method. For this problem, we can assume a good starting point for this problem would be $x_0 = 1$.

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
f(x) &= x^4 + x - 1 \\
f'(x) &= 4x^3 + 1 \qquad (2.2) \\
x_1 &= 1 - \frac{(1)^4 + (1) - 1}{4(1)^3 + 1} \\
x_1 &= \frac{4}{5}
\end{aligned}
$$

Then, we simply put the result we obtained from the first run of the Newton's Method and replace x_0 with x_1 to find an even better approximation.

$$
\begin{aligned}
x_2 &= \frac{4}{5} - \frac{\left(\frac{4}{5}\right)^4 + \left(\frac{4}{5}\right) - 1}{4\left(\frac{4}{5}\right)^3 + 1} \\
x_2 &\approx 0.731234
\end{aligned}
$$

This approximation is closer, but we can continue iterating to get an even better approximation. We ended up doing five iterations of the method, and our fifth iteration gave us $x_5 \approx 0.724492$. The solution from Wolfram Alpha (also technically an approximation since the real answer is irrational) is $x = 0.724491959$, which is an error of only $5.6 * 10^-8$.

3. Approximating a Reciprocal

Some computers cannot directly perform division. They are restricted to addition, subtraction and multiplication. Using Newton's Method, we can approximate the reciprocal of a number b using only these three operations. Recall that in order to analyze a problem by this method, we must express it in terms of root finding.

Theorem 3.1. *The reciprocal $\frac{1}{b}$ expressed as the root of a function.*

$$
\begin{aligned}
f(x) &= b - \frac{1}{x} \\
f'(x) &= \frac{1}{x^2}
\end{aligned}
$$

We can then use Newton's Method to approximate the location of this root, ending with a formula that contains no division whatsoever.

Theorem 3.2. *The recursive function to approximate the reciprocal $\frac{1}{b}$ without using division.*

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{b - \frac{1}{x_n}}{\frac{1}{{x_n}^2}} \\
&= x_n - \left(b - \frac{1}{x_n}\right)({x_n}^2) \qquad (3.1) \\
&= x_n - (b{x_n}^2 - x_n) \\
x_{n+1} &= x_n(2 - bx_n)
\end{aligned}
$$

Now we can test this by approximating the value of $\frac{1}{\pi}$ using $x_0 = 0.5$ and again using $x_0 = 0.7$.

Example 3.3. Approximating the reciprocal of π.
$b = \pi$ and $x_0 = 0.5$

$$\begin{aligned} x_1 &= 0.5(2 - 0.5\pi) \approx 0.214602 \\ x_2 &\approx 0.214602(2 - 0.214602\pi) \approx 0.284521 \\ x_3 &\approx 0.214602(2 - 0.214602\pi) \approx 0.314723 \\ x_4 &\approx 0.314723(2 - 0.314723\pi) \approx 0.318269 \end{aligned} \tag{3.2}$$

After only four iterations the result is already pretty close to the actual value $\frac{1}{\pi} \approx 0.318310$. But lets see what happens with an x_0 that's further away.
$b = \pi$ and $x_0 = 0.7$

$$\begin{aligned} x_1 &= 0.7(2 - 0.7\pi) \approx -0.13938 \\ x_2 &\approx -0.13938(2 - -0.13938\pi) \approx -0.339791 \\ x_3 &\approx -0.339791(2 - -0.3397912\pi) \approx -1.0423 \\ x_4 &\approx -1.0423(2 - -1.0423\pi) \approx -5.49759 \end{aligned} \tag{3.3}$$

We're getting nowhere fast with this.

There is a limit on how far our initial x_0 can be from the actual root before we end up overshooting and getting garbage results. This limit's nature can be known by building a function that plots how good our approximations are.

4. Error

Assuming we actually know the true location of the root we're searching for (or have a very good approximation already), we can build a function to determine the relative error of our approximation. The smaller it is, the better the approximation.

Theorem 4.1. *The general form of the formula to determine relative error.*

$$\frac{actual - approximate}{actual} = \frac{\alpha - x_n}{\alpha} = Rel(x_n)$$

We can specifically apply this to our reciprocal approximating function like so.

Example 4.2. The Relative Error function for $\frac{1}{b}$

$$\begin{aligned} Rel(x_n) &= \frac{\frac{1}{b} - x_n}{\frac{1}{b}} \\ &= \left(\frac{1}{b} - x_n\right)(b) \\ Rel(x_n) &= 1 - bx_n \end{aligned} \tag{4.1}$$

And likewise

$$Rel(x_{n+1}) = 1 - bx_{n+1} \tag{4.2}$$

However, $Rel(x_{n+1})$ is actually a quadratic function in terms of $Rel(x_n)$! We can prove this as follows.

Theorem 4.3. *$Rel(x_{n+1})$ is quadratic upon its last iteration.*
Assuming $Rel(x) = 1 - bx$ and $x_{n+1} = x_n(2 - bx_n)$

$$\begin{aligned} Rel(x_{n+1}) &= 1 - b(x_n(2 - bx_n))) \\ &= 1 - (bx_n(2 - bx_n)) \\ &= 1 - (2bx_n - (bx_n)^2 \\ &= (bx_n)^2 - 2bx_n + 1 \\ &= (bx_n - 1)^2 \\ &= (-1)(bx_n - 1)(-1)(bx_n - 1) \\ &= (1 - bx_n)^2 \\ Rel(x_{n+1}) &= Rel^2(x_n) \end{aligned} \tag{4.3}$$

This means that $Rel(x_0)$ must have an absolute value less than 1 or else our relative error will actually start INCREASING as we take more iterations and our approximations will get incrementally worse. So, exactly how far away can x_0 be in our case?

Example 4.4.

$$\begin{aligned} |Rel(x_0)| &< 1 \\ \pi x_0| &< 1 \end{aligned} \tag{4.4}$$

So lets find some solutions and see what region works with the original condition.

$$\begin{aligned} \sqrt{(1 - \pi x_0)^2} &= 1 \\ 1 - 2\pi x_0 + \pi^2 {x_0}^2 &= 1 \\ 2x - \pi {x_0}^2 &= 0 \\ (x)(2 - \pi x_0) &= 0 \end{aligned} \tag{4.5}$$

So our zeroes here are 0 and $\frac{2}{\pi}$. We know our region is actually between them from prior experiments, since 0.5 worked and $0 < 0.5 < \frac{2}{\pi}$. This leaves us with:

$$0 < x_0 < \frac{2}{\pi}$$

Every iteration, the relative error is equal to the square of the relative error of the previous iteration. That means as long as our initial estimate satisfies the condition, our estimates improve quadratically with each iteration. The closer we are, the closer we get with the next iteration. Of course this works the other way too; if we start out of bounds, our estimates get ridiculous proportionally to the square of the previous.

5. Examples with Newton's Method

There are some fun examples for which Newton's method produces interesting sequences of iterates. Here are two good examples of some such oddities.

Example 5.1. Estimating the root of $f(x) = \sqrt[3]{x}$ with $x_0 \neq 0$.

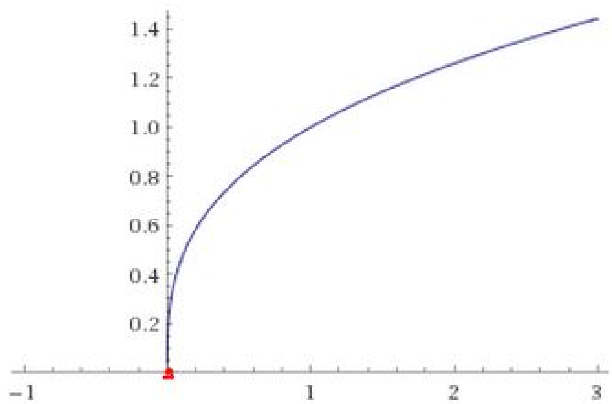

FIGURE 2. Graph of $f(x) = \sqrt[3]{x}$.

Obviously, from looking at the graph above we can tell that the x-intercept for this function is $x = 0$, but let's see what happens when we try to approximate it with Newton's Method. So we start off with $x_0 = 1$ and let us see what happens.

$$\begin{aligned} x_1 &= 1 - \frac{\sqrt[3]{1}}{3\sqrt[3]{(1)^2}} \\ x_1 &= 1 - \frac{1}{3} \\ x_1 &= \frac{2}{3} \end{aligned} \tag{5.1}$$

It seems that this is starting off okay. Let's see what happens with the next iteration.

$$\begin{aligned} x_2 &= \frac{2}{3} - \frac{\sqrt[3]{\frac{2}{3}}}{3\sqrt[3]{\left(\frac{2}{3}\right)^2}} \\ x_2 &\approx 0.2851 \end{aligned} \tag{5.2}$$

It seems to still be working. It continues to approach the real root of 0. Let's try another iteration.

$$\begin{aligned} x_3 &= 0.2851 - \frac{\sqrt[3]{0.2851}}{3\sqrt[3]{0.2851^2}} \\ x_3 &\approx -0.2214 \end{aligned} \tag{5.3}$$

Huh. That doesn't look right. It completely skipped over the actual answer of 0 and went into the negative somehow, even though the graph is only real in Quadrant I. Let's see what happens with one more iteration.

$$\begin{aligned} x_4 &= -0.2214 - \frac{\sqrt[3]{-0.2214}}{3\sqrt[3]{(-0.2214)^2}} \\ x_4 &\approx -0.496901 - .477181i \end{aligned} \tag{5.4}$$

And yet, it is not getting any closer to 0. As far as we can tell, the reason for this is that the derivative is undefined at $x = 0$, which breaks the approximation process.

Example 5.2. Estimating the root of $f(x) = x^3 - 2x + 2$

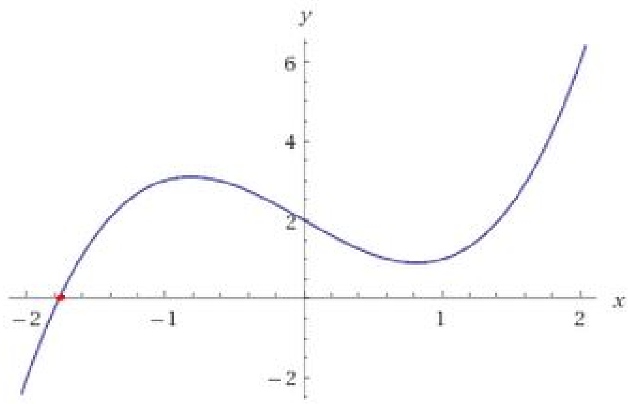

FIGURE 3. Graph of $f(x) = x^3 - 2x + 2$.

As is clearly visible from the graph, $f(x) = x^3 - 2x + 2$ has one real root near -2. Let's see what happens when we use Newton's Method with an initial guess that isn't relatively close to the point where the graph crosses the x-axis, such as $x_0 = 0$.

$$\begin{aligned} f'(x) &= 3x^2 - 2 \\ x_1 &= 0 - \frac{(0)^3 - 2(0) + 2}{3(0)^2 - 2} \\ x_1 &= 0 - \frac{2}{-2} \\ x_1 &= 1 \end{aligned} \tag{5.5}$$

That's not right. It seems to be moving in the opposite direction of the interception point. Let's observe what happens when we go through another iteration of the method.

$$\begin{aligned} x_2 &= 1 - \frac{(1)^3 - 2(1) + 2}{3(1)^2 - 2} \\ x_2 &= 1 - \frac{1}{1} \\ x_2 &= 0 \end{aligned} \tag{5.6}$$

Well that is strange. As we can see, no matter how many iterations we do, the answers will just alternate between $x = 0$ and $x = 1$. Let's see what happens when we choose and initial guess that is a little farther from our first guess, maybe $x_0 = .05$?

$$\begin{aligned} x_1 &= .05 - \frac{(.05)^3 - 2(.05) + 2}{3(.05)^2 - 2} \\ x_1 &\approx 1.0036 \end{aligned} \tag{5.7}$$

Yet it still seems to be acting as it did before with an initial guess of 0. Let's see if it follows the same pattern.

$$\begin{aligned} x_2 &= 1.0036 - \frac{(1.0036)^3 - 2(1.0036) + 2}{3(1.0036)^2 - 2} \\ x_2 &\approx 0.0212 \end{aligned} \tag{5.8}$$

It does. It turns out the next couple iterations are $x_3 = 1.0007$ and $x_4 = 0.0042$. We can observe that if we were to take even more iterations of the method, the values would get closer and closer to $x = 0$ and $x = 1$. It turns out that way for any initial guess in the interval $-0.1 < x < 0.1$.
Just for fun, let's see what happens when take an initial guess that is not even close to x-intercept, such as $x_0 = 5$.

$$\begin{aligned} x_1 &= 5 - \frac{(5)^3 - 2(5) + 2}{3(5)^2 - 2} \\ x_1 &= 5 - \frac{125 - 10 - 2}{75 - 2} \\ x_1 &= 5 - \frac{113}{73} \\ x_1 &\approx 3.4521 \end{aligned} \tag{5.9}$$

Well, it is getting closer, so let us do another iteration of Newton's Method.

$$\begin{aligned} x_2 &= 3.4521 - \frac{(3.4521)^3 - 2(3.4521) + 2}{3(3.4521)^2 - 2} \\ x_2 &\approx 2.3785 \end{aligned} \tag{5.10}$$

Still seems to be working. Let use the method once more.

$$\begin{aligned} x_3 &= 2.3785 - \frac{(2.3785)^3 - 2(2.3785) + 2}{3(2.3785)^2 - 2} \\ x_3 &\approx 1.6639 \end{aligned} \tag{5.11}$$

Slowly but surely it seems to be working. Let's see what happens after a few more iterations.

$$
\begin{aligned}
x_4 &= 1.6639 - \frac{(1.6639)^3 - 2(1.6639) + 2}{3(1.6639)^2 - 2} \\
x_4 &\approx 1.1439 \\
x_5 &= 1.1439 - \frac{(1.1439)^3 - 2(1.1439) + 2}{3(1.1439)^2 - 2} \\
x_5 &\approx 0.5160 \\
x_6 &= 0.5160 - \frac{(0.5160)^3 - 2(0.5160) + 2}{3(0.5160)^2 - 2} \\
x_6 &\approx 1.4362 \\
x_7 &= 1.4362 - \frac{(1.4362)^3 - 2(1.4362) + 2}{3(1.4362)^2 - 2} \\
x_7 &\approx 0.9372 \\
x_7 &= 0.9372 - \frac{(0.9372)^3 - 2(0.9372) + 2}{3(0.9372)^2 - 2} \\
x_7 &\approx -0.5569 \\
x_8 &= -0.5569 - \frac{(-0.5569)^3 - 2(-0.5569) + 2}{3(-0.5569)^2 - 2} \\
x_8 &\approx 2.1928 \\
x_9 &= 2.1928 - \frac{(2.1928)^3 - 2(2.1928) + 2}{3(2.1928)^2 - 2} \\
x_9 &\approx 1.5362
\end{aligned}
\tag{5.12}
$$

And yet it does not seem to be going anywhere. This example shows kind of the same back and forth pattern as before, and also supplements the fact that the initial guess should be relatively close to the actual x-intercept. Fascinating.

Conclusion

As we have observed from this report, Newton's Method can be a very efficient method of approximating a root of a function. It can also be used to approximate the intersection point of two functions. It seems that as long as the function you are approximating a solution for is relatively tame, and you choose an initial guess that is close to the intercept, it is very reliable. However, if you have a bad first guess or a function with a complicated derivative or slope, the method breaks and behaves rather oddly. All in all, as long as you know how to use it and know its limitations, then the Newton's Method is a great way to approximate roots.

References

1. Lake Ritter, (2014) Finding Roots: Newton's Method.

E-mail address: `ebanyash@spsu.edu`

E-mail address: `cbogolin@spsu.edu`

E-mail address: `dyoung7@spsu.edu`