Finding Roots: Newton's Method

Calculus I Project

The purpose of this project is to derive and analyze a method for solving equations. This method is iterative meaning that successive approximations to a solution are obtained with the intent that each new approximation is an improvement over the previous ones. Many mathematical problems can be restated in terms of **root finding**. (Recall that c is a root of a function $f(x)$ provided $f(c) = 0$.) Suppose for example that we wish to obtain a decimal approximation to the number π. Given a little knowledge of trigonometry, we can restate this problem in terms of roots as:

$$\text{Find the smallest positive number } x \text{ such that } \tan\left(\frac{x}{4}\right) - 1 = 0.$$

For an iterative method, we start with some initial value (perhaps an educated guess) x_0. We feed this into some sort of scheme (a formula) which produces a new value x_1. We then repeat the process to build a sequence $x_0, x_1, x_2, \ldots, x_n$ of values that are increasing closer to the true solution. Newton's method gives such a scheme based on the observation that for some functions, the root of a tangent line will be closer to the true root of a function.

Suppose that f is some function with true root α—note that α is an x-intercept. Let x_0 be some value assumed to be close to the root α, and let $L(x)$ be the tangent line to f at the point $(x_0, f(x_0))$ as shown in the figure. The number α is not known, and the goal is to determine its value. The number x_0 is an initial guess as to the value of α. We don't expect it to be correct, but we hope to use it to find a better estimate. Note that the x-intercept of the tangent line (x_1 in the figure) is closer to the

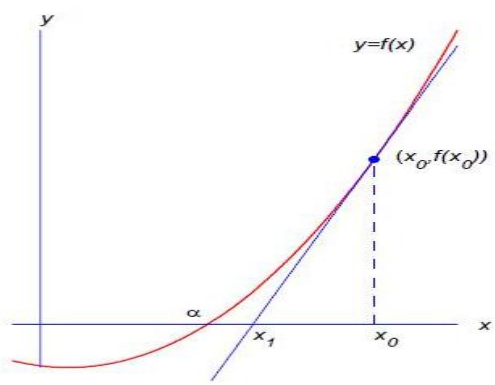

Figure 1: Graph of f showing true root α and tangent line at $(x_0, f(x_0))$.

true root α. For Newton's method, this new value x_1 is taken as the next approximation. Then the process is repeated by forming the tangent line to f at the new point $(x_1, f(x_1))$. The x-intercept of this new line is called x_2—the next, and hopefully better approximation to α. We continue to play this game until we're satisfied with the accuracy. In general, we stop the process when successive approximations are deemed close enough together—that is, when $|x_{n+1} - x_n| <$ some preset error tolerance.

Newton's method is widely used. When it works, it tends to work very quickly (requiring relatively few iterations). But it's not the only trick in the book, and it has its weaknesses. This project entails deriving the method as well as investigating its strengths and pit-falls.

Carry out the following activities.

A. Suppose that f is a function that has a root α that we wish to determine, and assume that f is at least a couple of times differentiable on an interval containing α. Let x_0 be an initial guess, and write the equation of the line $L(x)$ that is tangent to the graph of f at the point $(x_0, f(x_0))$. Use this to find a formula for x_1 the x-intercept of L in terms of x_0, $f(x_0)$, and $f'(x_0)$.

B. Repeat this process by writing the equation of a new tangent line L to the graph of f at the point $(x_1, f(x_1))$, and find a formula for x_2 the x-intercept of L in terms of x_1, $f(x_1)$ and $f'(x_1)$. Notice that the basic formula is the same one found before except with x_2 replacing x_1 and x_1 replacing x_0. Write this as a general formula for which the new iterate x_{n+1} is given in terms of the old iterate x_n, $f(x_n)$, and $f'(x_n)$.

C. The equation $\frac{1}{x} = 1 + x^3$ has one positive solution. Plot the two graphs $y = \frac{1}{x}$ and $y = 1 + x^3$ together on the same set of axis to demonstrate that there is one solution. Use Newton's method to find it correct to six decimal places. (Note that you will have to restate the problem as a root finding problem by defining an appropriate function $f(x)$ that has the solution as its root.)

D. Older (and some modern) computers do not directly perform the operation of division; they are restricted to computing with the operations of multiplication, addition, and subtraction. So, given a positive number b, its reciprocal must be computed indirectly. Note that for any such number b, the reciprocal $\frac{1}{b}$ is the true root of the function

$$f(x) = b - \frac{1}{x}.$$

Use Newton's method to find a sequence of approximations $x_0, x_1, \ldots, x_k$ to the reciprocal $\frac{1}{b}$ that **only requires the operations of $\times$, $+$, and/or $-$.** (Your iteration formula should simplify nicely.)

Now use this formula to find the reciprocal of π. Try two different cases: (1) use an initial guess of $x_0 = 0.5$, and find it to at least six decimal places; (2) use an initial guess of $x_0 = 0.7$ and comment on what you observe.

E. For a numerical scheme, we consider an error measure called the **relative error**. Suppose a true value is α and an estimate to this value is x_n. Then the relative error is

$$\text{Rel}(x_n) = \frac{\text{true value} - \text{approximate}}{\text{true value}} = \frac{\alpha - x_n}{\alpha}.$$

Consider the reciprocal finding process in part D. (with $\alpha = \frac{1}{b}$ for some positive b). Show that

$$\text{Rel}(x_n) = 1 - bx_n, \quad \text{and similarly} \quad \text{Rel}(x_{n+1}) = 1 - bx_{n+1}.$$

Use the formula for x_{n+1} to show that

$$\text{Rel}(x_{n+1}) = (\text{Rel}(x_n))^2 .$$

This puts a restriction on how good (close to α) the initial guess must be to ensure that the method works. Find a condition on x_0 or on $\text{Rel}(x_0)$ that will guarantee that the approximations get better with each iteration. Discuss what this means in terms of both the restriction on the method and how fast it works when it does work. (For example, if the relative error at some step is about 10^{-4}, how big would it be at the next step.)

F. There are fun examples for which Newton's method produces interesting sequences of iterates. Experiment with some of the following. Pick one (or two) and discuss the behavior of the method. You may wish to include graphical illustration of what you observe in addition to a table of interations.

i. The true root of $f(x) = \sqrt[3]{x}$ is clearly $\alpha = 0$. Newton's method can not be used to find this root for any choice of $x_0 \neq 0$.

ii. The equation $x^3 - 5x = 0$ has three solutions $0, \sqrt{5}$ and $-\sqrt{5}$. Consider Newton's method with $x_0 = 1$ or $x_0 = -1$.

iii. $f(x) = x^3 - 2x + 2$ has one real root close to -2. What happens with an initial gues of $x_0 = 0$? If an initial guess in the interval $-0.1 < x < 0.1$ is chosen, after several iterations, some interesting behavior is seen. What is perhaps more surprising is what happens if a really bad guess—e.g. $x_0 = 5$—is taken.