# November 11 Math 2306 sec 51 Fall 2015

## Section 7.3: Translation Theorems

**Theorem (translation in $s$)** Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number $a$
$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

Consequently $\mathscr{L}^{-1}\left\{F(s-a)\right\} = e^{at}f(t)$ if $\mathscr{L}^{-1}\left\{F(s)\right\} = f(t)$.

**Theorem (translation in $t$)** If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then
$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

## Show $\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{f(t)\}$

Recall $f(t-a)\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \ge a \end{cases}$

By definition

$$\mathscr{L}\{f(t-a)\mathcal{U}(t-a)\} = \int_0^\infty e^{-st} f(t-a)\mathcal{U}(t-a)\, dt$$

$$= \int_0^a e^{-st} f(t-a)\underbrace{\mathcal{U}(t-a)}^{=0} dt + \int_a^\infty e^{-st} f(t-a)\mathcal{U}(t-a)\, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt$$

$$= \int_0^\infty e^{-s(u+a)} f(u)\,du$$

Let $u = t - a$

$du = dt$

$t = u + a$

when $t = a$, $u = 0$

as $t \to \infty$, $u \to \infty$

$$= e^{-as} \int_0^\infty e^{-su} f(u)\,du$$

Use

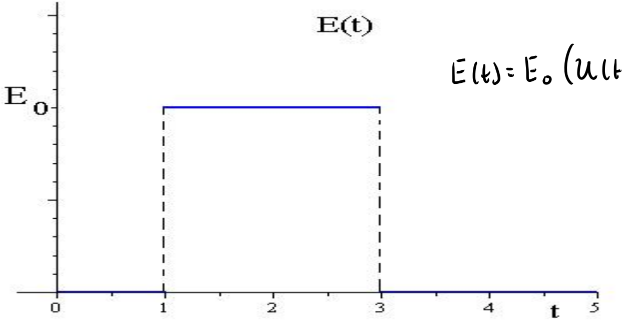
$e^{-s(u+a)} = e^{-su} \cdot e^{-sa}$

$$= e^{-as} \mathcal{L}\{f(t)\}$$

$$= e^{-as} F(s) \quad \text{if} \quad F(s) = \mathcal{L}\{f(t)\}$$

# Solve the IVP

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.




Note that

$$E(t) = E_0\left(\mathcal{U}(t-1) - \mathcal{U}(t-3)\right)$$

Basic Eqn is $L\frac{di}{dt} + Ri = E$

$$i' + 10i = E_0\left(\mathcal{U}(t-1) - \mathcal{U}(t-3)\right), \quad i(0) = 0$$

Let $I(s) = \mathcal{L}\{i(t)\}$

$$\mathcal{L}\{i' + 10i\} = \mathcal{L}\left\{E_0\left(\mathcal{U}(t-1) - \mathcal{U}(t-3)\right)\right\}$$

$$\mathcal{L}\{i'\} + 10\,\mathcal{L}\{i\} = E_0\,\mathcal{L}\{\mathcal{U}(t-1)\} - E_0\,\mathcal{L}\{\mathcal{U}(t-3)\}$$

$$sI(s) - i(0) + 10I(s) = \frac{E_0 e^{-s}}{s} - \frac{E_0 e^{-3s}}{s}$$

$$(s+10)I(s) = \frac{E_0 e^{-s}}{s} - \frac{E_0 e^{-3s}}{s}$$

$$I(s) = \frac{E_0 e^{-s}}{s(s+10)} - \frac{E_0 e^{-3s}}{s(s+10)}$$

Do a partial fraction decomp on

$$\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

$$1 = A(s+10) + Bs$$

set $s=0$, $1=10A$, $A=\frac{1}{10}$ set $s=-10$ $1=-10B$, $B=\frac{-1}{10}$

$$I(s) = \frac{1}{10}\frac{E_0 e^{-s}}{s} - \frac{1}{10}\frac{E_0 e^{-s}}{s+10} - \frac{1}{10}\frac{E_0 e^{-3s}}{s} + \frac{1}{10}\frac{E_0 e^{-3s}}{s+10}$$

We'll use

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)\,\mathcal{U}(t-a)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\} = e^{-10t}$$

$$i(t) = \mathcal{L}^{-1}\left\{I(s)\right\}$$

$$= \frac{E_0}{10}\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s}\right\} - \frac{E_0}{10}\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s+10}\right\}$$

$$-\frac{E_0}{10}\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s}\right\}+\frac{E_0}{10}\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s+10}\right\}$$

$$i(t)=\frac{E_0}{10}\mathcal{U}(t-1)-\frac{E_0}{10}e^{-10(t-1)}\mathcal{U}(t-1)-\frac{E_0}{10}\mathcal{U}(t-3)+\frac{E_0}{10}e^{-10(t-3)}\mathcal{U}(t-3)$$

In piecewise form

$$i(t)=\begin{cases}0, & 0\le t<1\\ \frac{E_0}{10}-\frac{E_0}{10}e^{-10(t-1)}, & 1\le t<3\\ -\frac{E_0}{10}e^{-10(t-1)}+\frac{E_0}{10}e^{-10(t-3)}, & t\ge 3\end{cases}$$

Note, $i(t)$ is continuous on $[0,\infty)$