November 11 Math 2306 sec 54 Fall 2015

Section 7.3: Translation Theorems

Theorem (translation in *s*) Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

Consequently $\mathscr{L}^{-1}\left\{F(s-a)\right\} = e^{at}f(t)$ if $\mathscr{L}^{-1}\left\{F(s)\right\} = f(t)$.

Theorem (translation in *t*) If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

Show $\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{f(t)\}$

Recall that $f(t-a)\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \ge a \end{cases}$

By definition

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = \int_0^\infty e^{-st} f(t-a)\,\mathcal{U}(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)\underbrace{\mathcal{U}(t-a)}_{=0}\,dt + \int_a^\infty e^{-st} f(t-a)\,\mathcal{U}(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt$$

Let $u = t-a$

so $t = u+a$

$du = dt$

when $t=a$, $u=0$

and

as $t \to \infty$, $u \to \infty$

$$= \int_0^\infty e^{-s(u+a)} f(u)\, du$$

Note

$e^{-s(u+a)} = e^{-su} \cdot e^{-sa}$

this is constant in u

$$= e^{-as} \int_0^\infty e^{-su} f(u)\, du$$

$$= e^{-as} \mathcal{L}\{f(t)\}$$

as expected

A Couple of Useful Results

Another formulation of this translation theorem is

(1) $\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$

since $g(t) = g(t-a+a)$

The inverse form of this tranlation theorem is

(2) $\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$

Evaluate the Laplace or Inverse Laplace Transform

(a) $\mathscr{L}\{2t\,\mathscr{U}(t-3)\} = e^{-3s}\mathscr{L}\{2(t+3)\}$

$$= e^{-3s}\mathscr{L}\{2t+6\}$$

$$= e^{-3s}\left(2\mathscr{L}\{t\}+6\mathscr{L}\{1\}\right)$$

$$= e^{-3s}\left(\frac{2}{s^2}+\frac{6}{s}\right)$$

(b) $\mathscr{L}\left\{\cos t\, \mathscr{U}\left(t-\frac{\pi}{2}\right)\right\}$ $= e^{-\frac{\pi}{2}s}\mathscr{L}\left\{\cos\left(t+\frac{\pi}{2}\right)\right\}$

$$= e^{-\frac{\pi}{2}s}\mathscr{L}\{-\sin t\}$$

$$= -e^{-\pi/2\, s}\frac{1}{s^2+1}$$

$$\cos\left(t+\frac{\pi}{2}\right) = \cos t\cos\pi/2 - \sin t\sin\pi/2 = -\sin t$$

(c) $\mathscr{L}^{-1}\left\{\frac{e^{-4s}}{s^3}\right\} = \mathscr{L}^{-1}\left\{ e^{-4s} \frac{1}{s^3} \right\}$

$= \frac{1}{2}(t-4)^2 \mathcal{U}(t-4)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2!} \frac{2!}{s^3}\right\}$

$= \frac{1}{2!} t^2$

(d) $\mathscr{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\} = \mathscr{L}^{-1}\left\{e^{-s}\,\frac{1}{s(s+1)}\right\}$

Decomp $\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{1}{s} - \frac{1}{s+1}$

$1 = A(s+1) + Bs$

set $s=0$ $\quad 1 = A$

$s=-1 \quad 1 = -B$

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s} - e^{-s}\frac{1}{s+1}\right\}$$

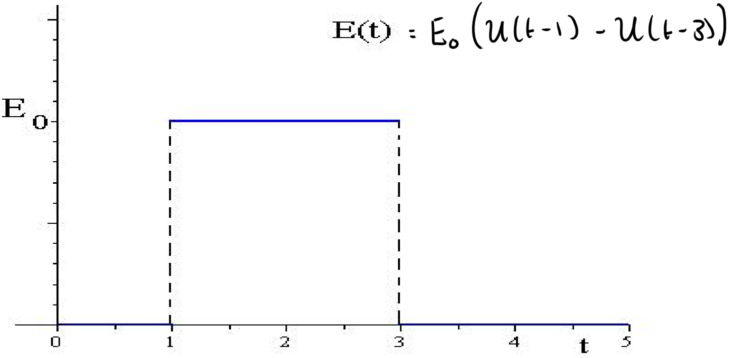
$$= \mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s+1}\right\}$$

$$= \mathcal{U}(t-1) - e^{-(t-1)}\mathcal{U}(t-1)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \qquad \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

Solve the IVP

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.



Basic Eqn $\quad L\frac{di}{dt} + Ri = E$

$$\frac{di}{dt} + 10i = E_0\mathcal{U}(t-1) - E_0\mathcal{U}(t-3), \quad i(0) = 0$$

let $\quad I(s) = \mathcal{L}\{i(t)\}$

$$\mathcal{L}\{i' + 10i\} = \mathcal{L}\{E_0\mathcal{U}(t-1) - E_0\mathcal{U}(t-3)\}$$

$$sI(s) - i(0) + 10I(s) = \frac{E_0}{s}e^{-s} - \frac{E_0}{s}e^{-3s}$$

$$(s+10)I(s) = \frac{E_0}{s}e^{-s} - \frac{E_0}{s}e^{-3s}$$

$$I(s) = \frac{E_0}{s(s+10)}e^{-s} - \frac{E_0}{s(s+10)}e^{-3s}$$

Do a decomp on $\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$

$$1 = A(s+10) + Bs$$

Set $s=0$ $\quad 1 = 10A \quad A = \frac{1}{10}$

$s=-10 \quad 1 = -10B \quad B = \frac{-1}{10}$

$$I(s) = \frac{E_0}{10} e^{-s} \frac{1}{s} - \frac{E_0}{10} e^{-s} \frac{1}{s+10} - \frac{E_0}{10} e^{-3s} \frac{1}{s} + \frac{E_0}{10} e^{-3s} \frac{1}{s+10}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad \mathscr{L}^{-1}\left\{\frac{1}{s+10}\right\} = e^{-10t}$$

$$i(t) = \mathscr{L}^{-1}\{I(s)\}$$

$$= \frac{E_0}{10} \mathscr{U}(t-1) - \frac{E_0}{10} e^{-10(t-1)} \mathscr{U}(t-1) - \frac{E_0}{10} \mathscr{U}(t-3) + \frac{E_0}{10} e^{-10(t-3)} \mathscr{U}(t-3).$$

$$i(t) = \begin{cases} 0, & 0 \le t < 1 \\ \frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)}, & 1 \le t < 3 \\ -\frac{E_0}{10} e^{-10(t-1)} + \frac{E_0}{10} e^{-10(t-3)}, & t \ge 3 \end{cases}$$