November 12 MATH 1113 sec. 51 Fall 2018

Section 6.6: Graphing Trigonometric Functions with Transformations

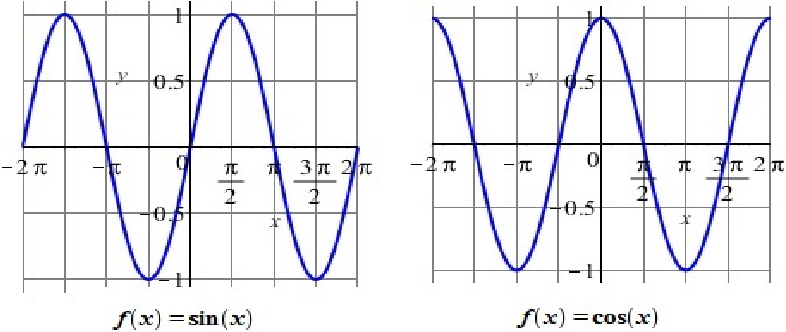


Figure: Recall the basic graphs of the six trigonometric functions. Sine and Cosine

Basic Trigonometric Graphs

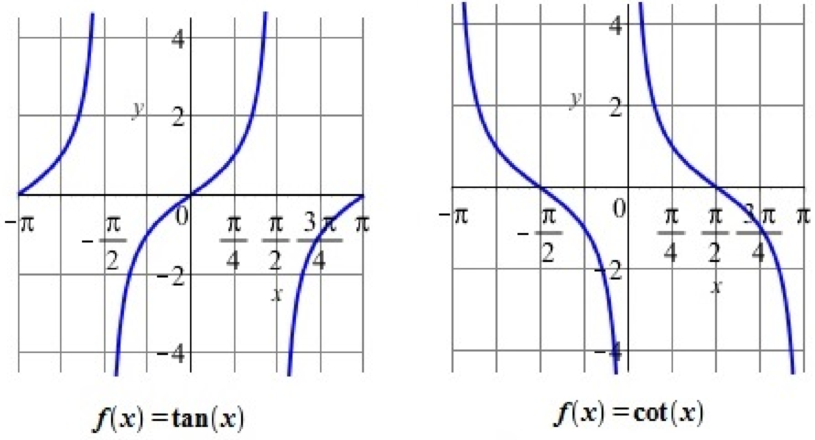
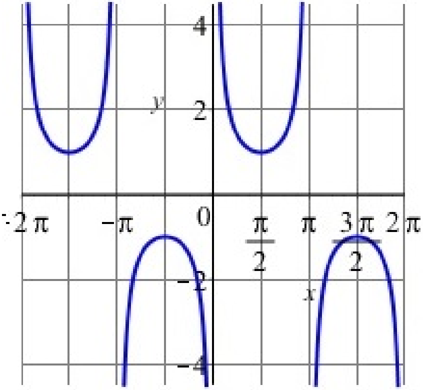
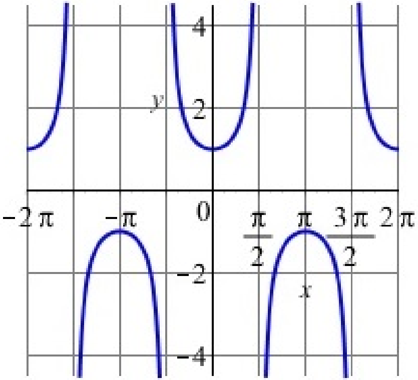


Figure: Recall the basic graphs of the six trigonometric functions. Tangent and Cotangent

Basic Trigonometric Graphs



$f(x) = \csc(x)$



$f(x) = \sec(x)$

Figure: Recall the basic graphs of the six trigonometric functions. Cosecant and Secant

Question

The fundamental period of $\sin x$ and $\cos x$ is

(a) π

(b) 2π

(c) $2n\pi$ for integers n

(d) there's no such thing as a *fundamental* period

Question

$\sin x = 0$ whenever x is

(a) $\frac{\pi}{2}$

(b) any negative number

(c) any real number

(d) $n\pi$ whenever n is an integer

Transformations on Sine and Cosine

Our goal is to graph functions of the form

$$f(x) = a\sin(bx - c) + d \quad \text{or} \quad f(x) = a\cos(bx - c) + d$$

Note: here we will be graphing points (x, y) on a curve $y = f(x)$.

Amplitude

Consider: $f(x) = a\sin(bx - c) + d$ or $f(x) = a\cos(bx - c) + d$

Definition: Let a be any nonzero real number. The **amplitude** of the function f defined above is the value $|a|$.

Recall that this is half the distance between the maximum and minimum values.

If $a < 0$ the graph is reflected in the x-axis. But the amplitude is still $|a|$.

Period

Consider: $f(x) = a\sin(bx - c) + d$ or $f(x) = a\cos(bx - c) + d$

Theorem: Let b be any positive real number. The **fundamental period** of the function f above is given by

$$T = \frac{2\pi}{b}.$$

Recall that the fundamental period of $\cos x$ and $\sin x$ is 2π.
Due to symmetry, we can always assume $b > 0$. Note for example

$$\sin(-3x + 2) = \sin(-(3x - 2)) = -\sin(3x - 2).$$

The period is **always positive.** The period in this example is $\frac{2\pi}{3}$.
Allowing b to be signed, the period would be written as

$$T = \frac{2\pi}{|b|}.$$

Period

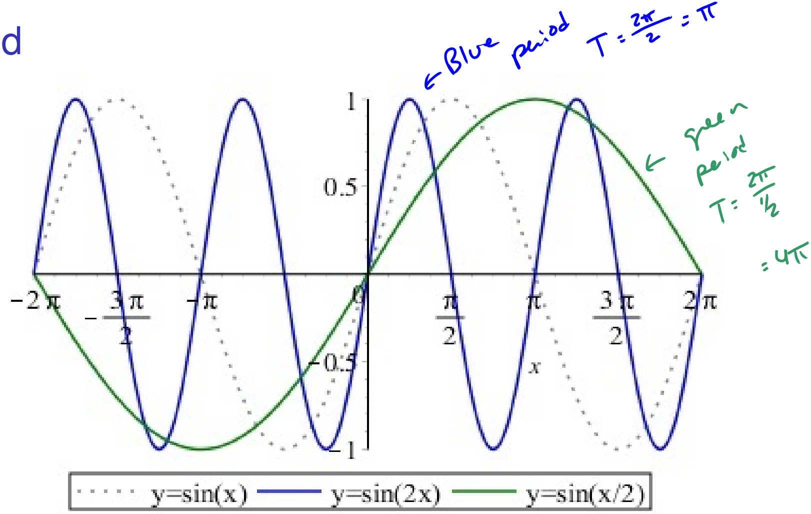


Figure: Comparisons with $b = 1/2, 1$, and 2. On the interval $-2\pi < x < 2\pi$ we obtain one ($b = 1/2$), two ($b = 1$) or four ($b = 2$) full cycles.

Period

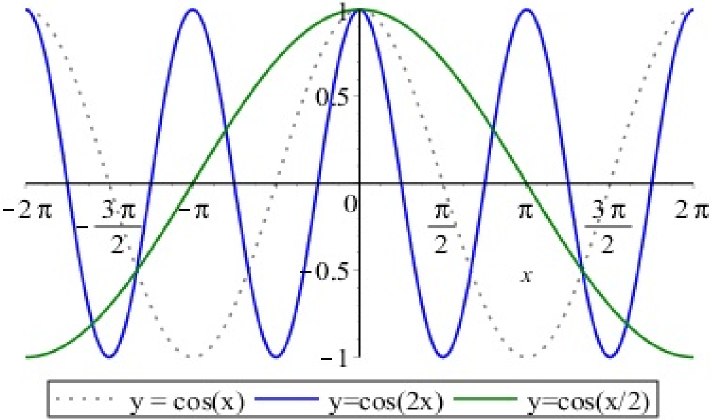


Figure: Comparisons with $b = 1/2, 1$, and 2. On the interval $-2\pi < x < 2\pi$ we obtain one ($b = 1/2$), two ($b = 1$) or four ($b = 2$) full cycles.

Example

Identify the period of each function.

(a) $f(x) = 3\sin(4x-2)+1$

$b = 4$ Period $T = \frac{2\pi}{4} = \frac{\pi}{2}$

(b) $f(x) = -5\sin\left(\frac{\pi x}{2}\right)+7$

$b = \frac{\pi}{2}$ period $T = \frac{2\pi}{\pi/2} = 4$

(c) $f(x) = 2-6\cos(2x+3)$

$b = 2$ period $T = \frac{2\pi}{2} = \pi$

Frequency

Consider: $f(x) = a\sin(bx - c) + d$ or $f(x) = a\cos(bx - c) + d$

Definition: The reciprocal of the period is called the **frequency**. That is

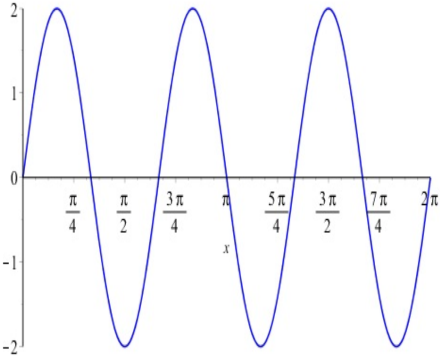
$$\text{frequency} = \frac{1}{T} = \frac{b}{2\pi}.$$

If x represents time, then

- the period tells us how much time is required for one full cycle, and
- the frequency tells us how many cycles occur in one time unit.

If $y = \cos(bx)$ (or $y = \sin(bx)$), then b the number of cycles occuring in an interval of length 2π.

Question



The figure shows $y = f(x)$ for $0 < x < 2\pi$. Which of the following is true?

(a) $f(x) = 2\sin\left(\frac{1}{3}x\right)$

(b) $f(x) = 2\sin(3x)$

(c) $f(x) = \frac{1}{2}\sin\left(\frac{1}{3}x\right)$

(d) $f(x) = \frac{1}{2}\sin(3x)$

Figure: Hint: Count the number of full cycles.

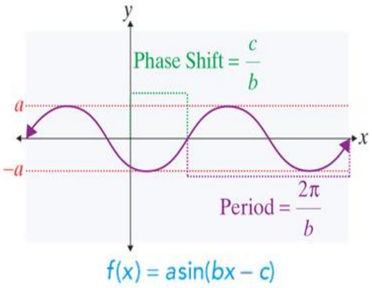
Phase Shift (horizontal shift)

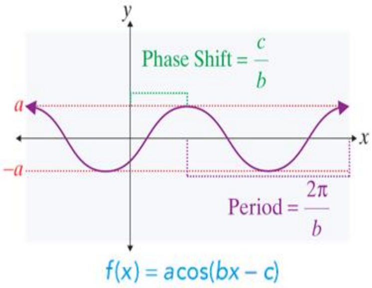
Consider: $f(x) = a\sin(bx - c) + d$ or $f(x) = a\cos(bx - c) + d$

Definition: A horizontal shift is called a **phase shift**. Again assuming that $b > 0$, the phase shift for f above is

$$\frac{|c|}{b} \quad \text{units}$$

to the right if $c > 0$ and to the left if $c < 0$.





Example

Identify the phase shift of each function. Determine if it is left or right.

(a) $f(x) = 3\sin(4x - 2) + 1$

phase shift $\frac{|c|}{b} = \frac{2}{4} = \frac{1}{2}$

$b = 4 \quad c = 2$

right

(b) $f(x) = -5\sin\left(\frac{\pi x}{2}\right) + 7$

No phase shift

$b = \frac{\pi}{2} \quad c = 0$

(c) $f(x) = 2 - 6\cos(2x + 3)$

phase shift $\frac{|c|}{b} = \frac{|-3|}{2} = \frac{3}{2}$

to the left

$b = 2 \quad c = -3$

Question

Let $f(x) = -3\sin\left(\frac{\pi x}{4} - \frac{1}{3}\right) + 1$. Then which of the following it true regarding the phase shift.

(a) The phase shift is $\frac{4}{3\pi}$, the graph is shifted to the left.

(b) The phase shift is $\frac{4}{3\pi}$, the graph is shifted to the right.

$\frac{|c|}{b} = \frac{1/3}{\pi/4} = \frac{4}{3\pi}$ $c > 0$

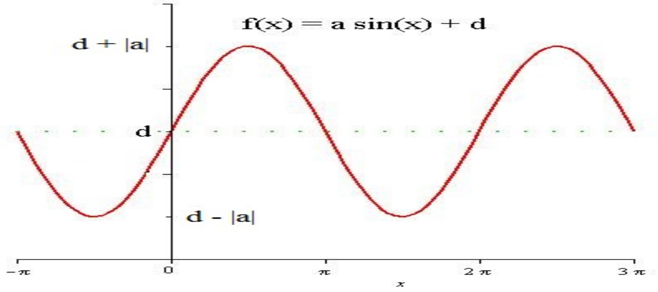
(c) The phase shift is $\frac{3\pi}{4}$, the graph is shifted to the left.

(d) The phase shift is $\frac{3\pi}{4}$, the graph is shifted to the right.

Vertical Shift

Consider: $f(x) = a\sin(bx - c) + d$ or $f(x) = a\cos(bx - c) + d$

Definition: If d is a nonzero number, then the function f has a **vertical shift** of $|d|$ units up if $d > 0$ and down if $d < 0$.



Example

Identify the vertical shift of each function. Determine if it is up or down.

(a) $f(x) = 3\sin(4x-2)+1$ Vertical shift 1 unit up

$d = 1$

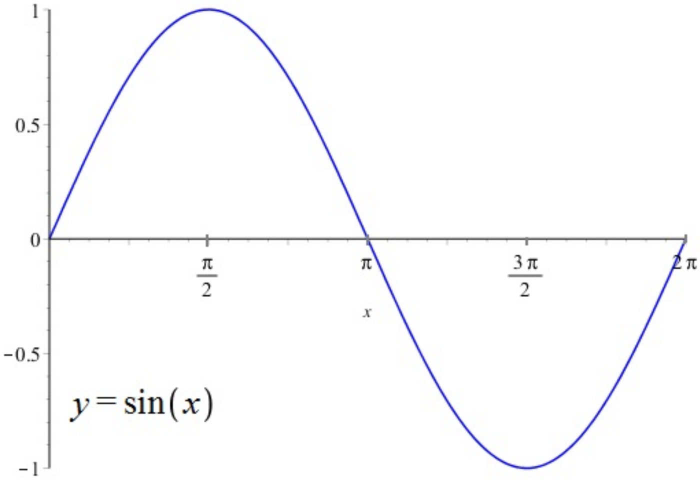
(b) $f(x) = -5\sin\left(\frac{\pi x}{2}\right)-7$ shift 7 units down

$d = -7$

(c) $f(x) = 2-6\cos(2x+3)$ shift 2 units up

$d = 2$

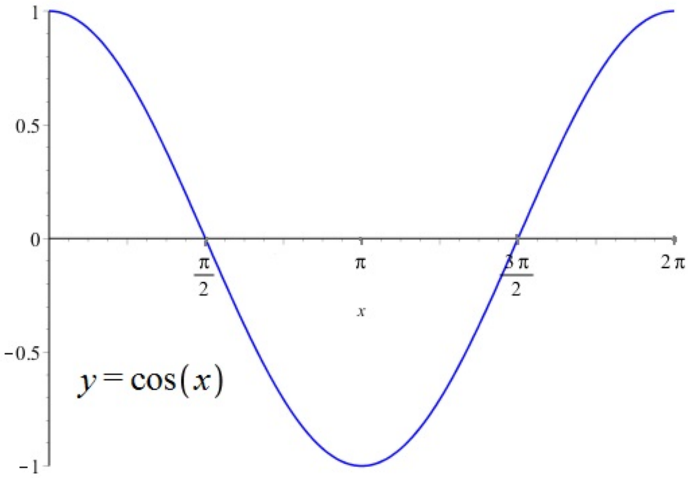
Parent Plots



The period can be divided into four equal segments.

For the sine function x-int $\rightarrow$ max $\rightarrow$ x-int $\rightarrow$ min $\rightarrow$ x-int

Parent Plots



The period can be divided into four equal segments.

For the cosine function max → x-int → min → x-int → max

Pulling it all Together!

Plot two full periods of the function $f(x) = a\sin(bx - c) + d$ (or $f(x) = a\cos(bx - c) + d$). Carry out each of the following steps:

- Identify the amplitude and determine if there is an x-axis reflection.
- Identify the period. Find the length of one fourth of the period.
- Identify any phase shift with its direction. Identify end points and points that divide the period into four equal parts.
- Identify any vertical shift with its direction.
- Use the basic plot of $y = \sin x$ or $y = \cos x$ to get the profile.

$$f(x) = 2 - 4\cos\left(\pi x - \frac{\pi}{2}\right)$$

Identify the amplitude and vertical shift. Find the maximum and minimum values and determine if there is a reflection in the horizontal.

Amplitude $A = |a| = |-4| = 4$

Vertical shift 2 units up

$d = 2$

There is a reflection in the horizontal.

Max value is $2 + |-4| = 6$

Min value is $2 - |-4| = -2$

$$f(x) = 2 - 4\cos\left(\pi x - \frac{\pi}{2}\right)$$

Find the period and phase shift with direction.

Period $T = \frac{2\pi}{b} = \frac{2\pi}{\pi} = 2$

Phase shift $\frac{|c|}{b} = \frac{\pi/2}{\pi} = \frac{1}{2}$

Shift is $\frac{1}{2}$ unit to the right

$f(x) = 2 - 4\cos\left(\pi x - \frac{\pi}{2}\right)$

Identify the begining and end of one period, and divide it into fourths to determine where major points on the graph are (max/mins and translated x-intercepts)

The graph should start $\frac{1}{2}$ units to the right of the y-axis.

A period for $\cos x$ starts @ 0 and ends @ 2π

$\pi x - \frac{\pi}{2} = 0 \Rightarrow \pi x = \frac{\pi}{2} \Rightarrow x = \frac{1}{2}$ start

$\pi x - \frac{\pi}{2} = 2\pi \Rightarrow \pi x = \frac{5\pi}{2} \Rightarrow x = \frac{5}{2}$ finish

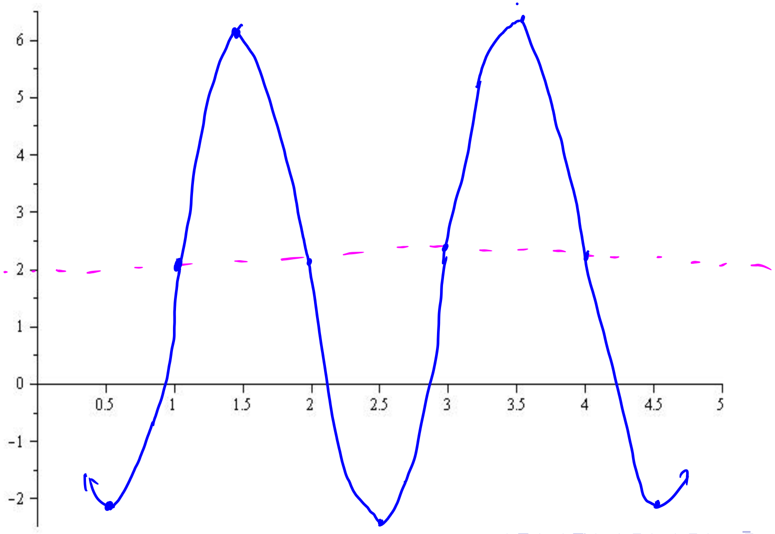
→ 2 units total

$\frac{1}{4}T = \frac{1}{4}(2) = \frac{1}{2}$ key x-values

$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
min	int.	max	int	min

$f(x) = 2 - 4\cos\left(\pi x - \frac{\pi}{2}\right)$

Plot two periods of its graph.



Section 7.4: Inverse Trigonometric Functions

Question: If someone asks "what is the sine of $\frac{\pi}{6}$?" we can respond with the answer (from memory or perhaps using a calculator) "$\frac{1}{2}$". What if the question is reversed? What if someone asks

"What angle has a sine value of $\frac{1}{2}$?"

$\sin\frac{\pi}{6} = \frac{1}{2}$

One such angle is $\frac{\pi}{6}$, another is $\frac{5\pi}{6}$

Restricting the Domain of sin(x)

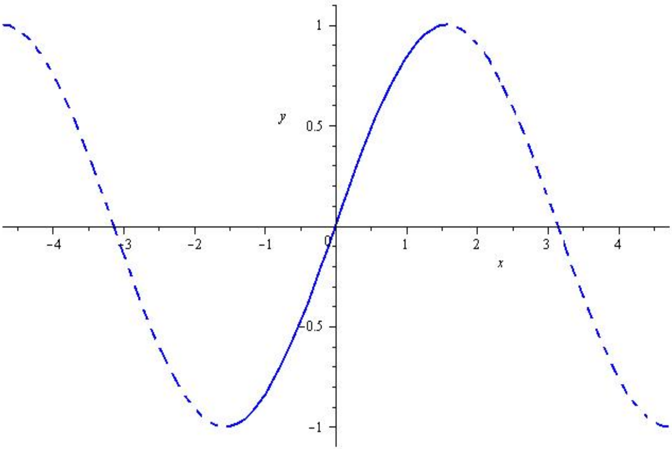


Figure: To define an inverse sine function, we start by restricting the domain of sin(x) to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The Inverse Sine Function (a.k.a. arcsine function)

Definition: For x in the interval $[-1, 1]$ the inverse sine of x is denoted by either

$$\sin^{-1}(x) \quad \text{or} \quad \arcsin(x)$$

and is defined by the relationship

$$y = \sin^{-1}(x) \quad \iff \quad x = \sin(y) \quad \text{where} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

The Domain of the Inverse Sine is $-1 \le x \le 1$**.**

The Range of the Inverse Sine is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$**.**

Notation Warning!

Caution: We must remember not to confuse the superscript -1 notation with reciprocal. That is

$$\sin^{-1}(x) \neq \frac{1}{\sin(x)}.$$

If we want to indicate a reciprocal, we should use parentheses or trigonometric identities

$$\frac{1}{\sin(x)} = (\sin(x))^{-1} \quad \text{or write} \quad \frac{1}{\sin(x)} = \csc(x).$$

Some Inverse Sine Values

We can build a table of some inverse sine values by using our knowledge of the sine function.

x	$\sin(x)$
$-\frac{\pi}{2}$	-1
$-\frac{\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$-\frac{\pi}{6}$	$-\frac{1}{2}$
0	0
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

x	$\sin^{-1}(x)$
-1	$-\frac{\pi}{2}$
$-\frac{1}{\sqrt{2}}$	$-\frac{\pi}{4}$
$-\frac{1}{2}$	$-\frac{\pi}{6}$
0	0
$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$
1	$\frac{\pi}{2}$

Conceptual Definition[1]

We can think of the inverse sine function in the following way:

$\sin^{-1}(x)$ **is the *angle* between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x.**

[1] We want to consider $f(x) = \sin^{-1} x$ as a real valued function of a real variable without necessary reference to angles, triangles, or circles. But the above is a **very useful** conceptual device for working with and evaluating this function.