November 12 Math 2306 sec. 53 Fall 2018

Section 16: Laplace Transforms of Derivatives and IVPs

Solve the initial value problem.

$$y''+4y'+4y = te^{-2t} \quad y(0) = 1, y'(0) = 0$$

We got started on this problem on Friday, but we ran out of time. We took the transform of both sides using the convention $Y(s) = \mathscr{L}\{y\}$

$$\begin{aligned}
\mathscr{L}\{y''+4y'+4y\} &= \mathscr{L}\left\{te^{-2t}\right\}\\
\mathscr{L}\{y''\}+4\mathscr{L}\{y'\}+4\mathscr{L}\{y\} &= \frac{1}{(s+2)^2}\\
s^2Y(s)-sy(0)-y'(0)+4(sY(s)-y(0))+4Y(s) &= \frac{1}{(s+2)^2}\\
(s^2+4s+4)Y(s)-s-4 &= \frac{1}{(s+2)^2}
\end{aligned}$$

The last line includes use of the given initial conditions.

$$(s^2+4s+4)\,Y(s) = \frac{1}{(s+2)^2} + s+4$$

$$Y(s) = \frac{1}{(s+2)^2(s^2+4s+4)} + \frac{s+4}{s^2+4s+4}$$

Note $s^2+4s+4 = (s+2)^2$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+4}{(s+2)^2}$$

Note $s+4 = s+2\ +2$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+2\ +2}{(s+2)^2}$$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2}$$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

form $\frac{1}{s^4}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3!}\,\frac{3!}{s^4}\right\}$$

$$= \frac{1}{3!}\,t^3$$

form $\frac{1}{s^2}$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 2t$$

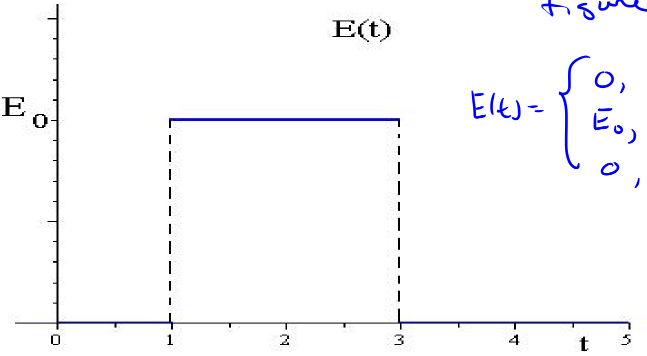
Finally we get $y(t) = \mathscr{L}^{-1}\{Y(s)\}$

$$y(t) = \mathscr{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathscr{L}^{-1}\left\{\frac{2}{(s+2)^2}\right\}$$

$$= \frac{1}{3!}t^3 e^{-2t} + e^{-2t} + 2te^{-2t}$$

Solve the IVP

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.



From the figure

$$E(t) = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

LR Circuit Example

In terms of step functions

$$E(t) = 0 - 0\mathcal{U}(t-1) + E_0\mathcal{U}(t-1) - E_0\mathcal{U}(t-3) + 0\mathcal{U}(t-3)$$

$$= E_0\mathcal{U}(t-1) - E_0\mathcal{U}(t-3)$$

The IVP to solve is

$$\frac{di}{dt} + 10i = E_0\mathcal{U}(t-1) - E_0\mathcal{U}(t-3)$$

$$i(0) = 0$$

* The basic equation is

$$L\frac{di}{dt} + Ri = E$$

We'll finish this next time!

Let's do one *together*

Solve the IVP using Laplace transforms.

$$y''+9y=36, \quad y(0)=2 \quad y'(0)=-1$$

Take the Laplace transform of both sides applying the transform property for the derivatives, and substituting in the initial conditions.

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{36\}$$

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{36}{s}$$

$$s^2Y(s) - 2s + 1 + 9Y(s) = \frac{36}{s}$$

$$y'' + 9y = 36, \quad y(0) = 2 \quad y'(0) = -1$$

At this point we have

$$s^2Y(s) - 2s + 1 + 9Y(s) = \frac{36}{s}$$

Now isolate $Y(s)$

$$(s^2+9)Y(s) = \frac{36}{s} + 2s - 1$$

$$Y(s) = \frac{36}{s(s^2+9)} + \frac{2s-1}{s^2+9}$$

$$= \frac{36}{s(s^2+9)} + \frac{2s}{s^2+9} - \frac{1}{s^2+9}$$

$$y'' + 9y = 36, \quad y(0) = 2 \quad y'(0) = -1$$

$$Y(s) = \frac{36}{s(s^2+9)} + \frac{2s}{s^2+9} - \frac{1}{s^2+9}$$

Do any necessary partial fraction decomposition, and collect any like terms.

$$\frac{36}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

$A = 4$

$B = -4$

$C = 0$

$$36 = A(s^2+9) + (Bs+C)s$$

$$= (A+B)s^2 + Cs + 9A$$

$$Y(s) = \frac{4}{s} - \frac{4s}{s^2+9} + \frac{2s}{s^2+9} - \frac{1}{s^2+9}$$

$$= \frac{4}{s} - \frac{2s}{s^2+9} - \frac{1}{s^2+9}$$

$$y'' + 9y = 36, \quad y(0) = 2 \quad y'(0) = -1$$

At this point, we have

$$Y(s) = \frac{4}{s} - \frac{2s}{s^2+9} - \frac{1}{s^2+9}$$

Finally take the inverse transform to find the solution $y(t)$ to the IVP.

$$\boxed{y(t) = 4 - 2\cos(3t) - \frac{1}{3}\sin(3t)}$$

Verify: $y' = 6\sin(3t) - \cos(3t)$

$y'' = 18\cos(3t) + 3\sin(3t)$

$y(0) = 4 - 2 - 0 = 2$

$y'(0) = 6\cdot 0 - 1 = -1$

$$y'' + 9y = 18\cos(3t) + 3\sin(3t) + 9\left(4 - 2\cos(3t) - \frac{1}{3}\sin(3t)\right)$$

$$= 18\cos(3t) + 3\sin(3t) + 36 - 18\cos(3t) - 3\sin(3t) = 36$$