

### Section 11.1: (Brief Overview of Inner Product and Orthogonality)

Suppose two functions  $f$  and  $g$  are integrable on the interval  $[a, b]$ . We define the **inner product** of  $f$  and  $g$  on  $[a, b]$  as

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx.$$

We say that  $f$  and  $g$  are **orthogonal** on  $[a, b]$  if

$$\langle f, g \rangle = 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

# Properties of an Inner Product

Let  $f$ ,  $g$ , and  $h$  be integrable functions on the appropriate interval and let  $c$  be any real number. The following hold

(i)  $\langle f, g \rangle = \langle g, f \rangle$

(ii)  $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$

(iii)  $\langle cf, g \rangle = c \langle f, g \rangle$

(iv)  $\langle f, f \rangle \geq 0$  and  $\langle f, f \rangle = 0$  if and only if  $f = 0$

## Orthogonal Set

A set of functions  $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$  is said to be **orthogonal** on an interval  $[a, b]$  if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x) \phi_n(x) dx = 0 \quad \text{whenever} \quad m \neq n.$$

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Note that any function  $\phi(x)$  that is not identically zero will satisfy

$$\langle \phi, \phi \rangle = \int_a^b \phi^2(x) dx > 0.$$

Hence we define the **square norm** of  $\phi$  (on  $[a, b]$ ) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x) dx}.$$

# An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \dots, \sin x, \sin 2x, \sin 3x, \dots\} \text{ on } [-\pi, \pi].$$

Show that  $\phi_0(x) = 1$ , is orthogonal to every function of the form  $\cos nx$  or  $\sin mx$  for all  $n \geq 1$  or  $m \geq 1$  on  $[-\pi, \pi]$ .

$$\begin{aligned}\langle 1, \cos(nx) \rangle &= \int_{-\pi}^{\pi} 1 \cdot \cos(nx) dx = \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi} \\ &= \frac{1}{n} \sin(n\pi) - \frac{1}{n} \sin(-n\pi)\end{aligned}$$

Recall

$$\sin(n\pi) = 0 \quad \text{for all integers}$$

$$= 0 - 0 = 0$$

$n$

So 1 is orthogonal to  $\cos(nx)$  for all  $n$

$$\langle 1, \sin(mx) \rangle = \int_{-\pi}^{\pi} 1 \cdot \sin(mx) dx = 0$$

since  $\sin(mx)$  is odd and  $[-\pi, \pi]$   
is a symmetric interval.

So 1 and  $\sin(mx)$  are orthogonal  
for each integer  $m$ .

## An Orthogonal Set of Functions continued...

Use the fact that  $\sin mx$  is an odd function and  $\cos nx$  is an even function for any choice of  $m$  and  $n$  to show that

$$\int_{-\pi}^{\pi} \cos nx \sin mx dx = 0 \quad \text{for all } m, n \geq 1.$$

Note  $\cos(n(-x)) \sin(m(-x)) = \cos(nx) [-\sin(mx)]$

$$= -\cos(nx) \sin(mx)$$

so  $\cos(nx) \sin(mx)$  is odd

Hence

$$\langle \cos(nx), \sin(mx) \rangle = \int_{-\pi}^{\pi} \cos(nx) \sin(mx) dx = 0 \Rightarrow \cos(nx) \sin(mx)$$

are orthogonal

## An Orthogonal Set of Functions continued...

Use the identity

$$\cos nx \cos mx = \frac{1}{2}(\cos((n+m)x) + \cos((n-m)x))$$

to show that

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = 0 \quad \text{whenever } n \neq m.$$

$$\begin{aligned}\langle \cos(nx), \cos(mx) \rangle &= \int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos((n+m)x) + \cos((n-m)x)] dx \\ &= \frac{1}{2} \left[ \frac{1}{n+m} \sin((n+m)x) + \frac{1}{n-m} \sin((n-m)x) \right] \Big|_{-\pi}^{\pi}\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \left[ \frac{1}{n+m} \sin((n+m)\pi) + \frac{1}{n-m} \sin((n-m)\pi) \right. \\
 &\quad \left. - \frac{1}{n+m} \sin(-(n+m)\pi) - \frac{1}{n-m} \sin(-(n-m)\pi \right] \\
 &= 0
 \end{aligned}$$

Hence  $\cos(nx)$  and  $\cos(mx)$  are  
orthogonal if  $n \neq m$ .

Evaluate

$$\int_{-\pi}^{\pi} \cos nx \cos nx dx = \int_{-\pi}^{\pi} \cos^2(nx) dx$$

$$= \int_{-\pi}^{\pi} \left( \frac{1}{2} + \frac{1}{2} \cos(2nx) \right) dx$$

$$= \frac{x}{2} \Big|_{-\pi}^{\pi} + \frac{1}{4n} \sin(2nx) \Big|_{-\pi}^{\pi} = \frac{\pi}{2} - \frac{(-\pi)}{2} + \frac{1}{4n} \sin(2n\pi) - \frac{1}{4n} \sin(-2n\pi)$$

$$= \pi$$

$$\text{Note } |\cos(nx)| = \sqrt{\pi}$$