# November 13 Math 2306 sec. 53 Fall 2019

## Section 16: Laplace Transforms of Derivatives and IVPs

Suppose $f$ has a Laplace transform[1] and that $f$ is differentiable on $[0,\infty)$. Obtain an expression for the Laplace tranform of $f'(t)$.

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)\,dt$$

Int. by parts

$u = e^{-st}$ $\quad du = -se^{-st}\,dt$

$v = f(t)$ $\quad dv = f'(t)\,dt$

$$= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty -se^{-st}f(t)\,dt$$

$$= 0 - e^{0}f(0) + s\int_0^\infty e^{-st}f(t)\,dt$$

$$= s\mathscr{L}\{f(t)\} - f(0)$$

$e^{-st}f(t) \to 0$ as $t \to \infty$ for $s > c$

[1] Assume $f$ is of exponential order $c$ for some $c$.

# Transforms of Derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, we have $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of $f$.

For example

$$\mathscr{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0)$$

$$= s\left(sF(s) - f(0)\right) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

# Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives $y'$, $y''$ and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

# Differential Equation

For constants $a$, $b$, and $c$, take the Laplace transform of both sides of the equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

$y(t)$ is the solution to the IVP; we'll find its Laplace transform. Let $\mathscr{L}\{y(t)\} = Y(s)$ and $\mathscr{L}\{g(t)\} = G(s)$.

$$\mathscr{L}\{ay'' + by' + cy\} = \mathscr{L}\{g(t)\}.$$

$$a\mathscr{L}\{y''\} + b\mathscr{L}\{y'\} + c\mathscr{L}\{y\} = \mathscr{L}\{g\}$$

$$a\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) = G(s)$$

We'll isolate $Y(s)$ using algebra.

$$(as^2+bs+c)Y(s) - say(0) - ay'(0) - by(0) = G(s)$$

$$(as^2+bs+c)\,Y(s) - say_0 - ay_1 - by_0 = G(s)$$

$$\underbrace{(as^2+bs+c)}_{\text{characteristic polynomial for the ODE}}Y(s) = say_0 + ay_1 + by_0 + G(s)$$

$$Y(s) = \frac{say_0+ay_1+by_0}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c}$$

The solution to the IVP

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}.$$

# Solving IVPs

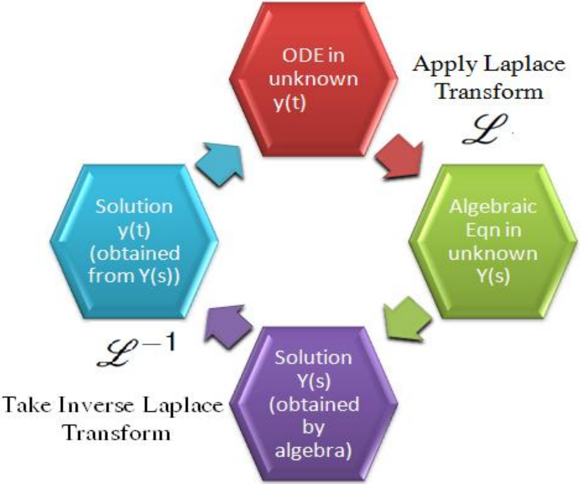


Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

# General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where $Q$ is a polynomial with coefficients determined by the initial conditions, $G$ is the Laplace transform of $g(t)$ and $P$ is the **characteristic polynomial** of the original equation.

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response.**