November 13 Math 2306 sec 54 Fall 2015

Section 11.1: (Brief Overview of Inner Product and Orthogonality)

Suppose two functions f and g are integrable on the interval $[a, b]$. We define the **inner product** of f and g on $[a, b]$ as

$$< f, g >= \int_a^b f(x)g(x)\, dx.$$

We say that f and g are **orthogonal** on $[a, b]$ if

$$< f, g >= 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

Properties of an Inner Product

Let f, g, and h be integrable functions on the appropriate interval and let c be any real number. The following hold

(i) $< f, g >=< g, f >$

(ii) $< f, g + h >=< f, g > + < f, h >$

(iii) $< cf, g >= c < f, g >$

(iv) $< f, f >\geq 0$ and $< f, f >= 0$ if and only if $f = 0$

Orthogonal Set

A set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$< \phi_m, \phi_n >= \int_a^b \phi_m(x)\phi_n(x)\,dx = 0 \quad \text{whenever} \quad m \neq n.$$

Note that any function $\phi(x)$ that is not identically zero will satisfy

$$< \phi, \phi >= \int_a^b \phi^2(x)\,dx > 0.$$

Hence we define the **square norm** of ϕ (on $[a, b]$) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x)\,dx}.$$

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

Show that $\phi_0(x) = 1$, is orthogonal to every function of the form $\cos nx$ or $\sin mx$ for all $n \geq 1$ or $m \geq 1$ on $[-\pi, \pi]$.

$$\langle 1, \cos(nx) \rangle = \int_{-\pi}^{\pi} 1 \cdot \cos(nx)\,dx = \frac{1}{n}\sin(nx)\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{n}\sin(n\pi) - \frac{1}{n}\sin(-n\pi) = 0 - 0 = 0$$

$*$ $\sin(n\pi) = 0$ for every integer n

Hence 1 and $\cos(nx)$ are orthogonal for any $n \geq 1$.

$$\langle 1, \sin(mx)\rangle = \int_{-\pi}^{\pi} 1\cdot \sin(mx)\,dx = \frac{-1}{m}\cos(mx)\Big|_{-\pi}^{\pi}$$

$$= \frac{-1}{m}\cos(m\pi) - \frac{-1}{m}\cos(-m\pi)$$

$$= \frac{-1}{m}\cos(m\pi) + \frac{1}{m}\cos(m\pi) = 0$$

* Recall

$$\cos(-\theta) = \cos\theta$$

Also

$$\cos(m\pi) = \begin{cases} -1, & m \text{ odd} \\ 1, & m \text{ even} \end{cases}$$

$$= (-1)^m$$

So 1 and $\sin(mx)$ are orthogonal for all $m \geq 1$.

An Orthogonal Set of Functions continued...

Use the fact that $\sin mx$ is an odd function and $\cos nx$ is an even function for any choice of m and n to show that

$$\int_{-\pi}^{\pi} \cos nx \, \sin mx \, dx = 0 \quad \text{for all} \quad m, n \geq 1.$$

Note

$\cos(-nx)\sin(-mx) = \cos(nx)\left[-\sin(mx)\right]$

$= -\cos(nx)\sin(mx)$

Hence the product is odd

$\langle \cos(nx), \sin(mx) \rangle$

$= \int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0$ by symmetry

So $\cos(nx)$ and $\sin(mx)$ are orthogonal for all $n, m \geq 1$