November 14 MATH 1113 sec. 51 Fall 2018

Section 7.4: Inverse Trigonometric Functions

Definition (The Inverse Sine Function): For x in the interval $[-1,1]$ the inverse sine of x is denoted by either

$$\sin^{-1}(x) \quad \text{or} \quad \arcsin(x)$$

quad I and IV

and is defined by the relationship

$$y=\sin^{-1}(x) \iff x=\sin(y) \quad \text{where} \quad -\frac{\pi}{2}\le y\le\frac{\pi}{2}.$$

The inverse sine function is also called the *arcsine function*.

Conceptual Definition[1]

We can think of the inverse sine function in the following way:

$\sin^{-1}(x)$ **is the *angle* between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x.**

[1] We want to consider $f(x) = \sin^{-1} x$ as a real valued function of a real variable without necessary reference to angles, triangles, or circles. But the above is a **very useful** conceptual device for working with and evaluating this function.

Example

Evaluate each expression exactly.

(a) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ because $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

and $\frac{-\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2}$

(b) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{-\pi}{3}$ since $\sin\left(\frac{-\pi}{3}\right) = \frac{-\sqrt{3}}{2}$

and

$\frac{-\pi}{2} \leq -\frac{\pi}{3} \leq \frac{\pi}{2}$

Question

Recall that $\sin(x) = 1$ for $x = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots, \frac{\pi}{2} + 2k\pi$, for integers k. And the range of the inverse sine is $[-\frac{\pi}{2}, \frac{\pi}{2}]$

The value of $\sin^{-1} 1$ is

(a) $\dfrac{\pi}{2}$

(b) $\dfrac{\pi}{2}$ and $\dfrac{5\pi}{2}$

(c) 0

(d) 0 and π

The Graph of the Arcsine

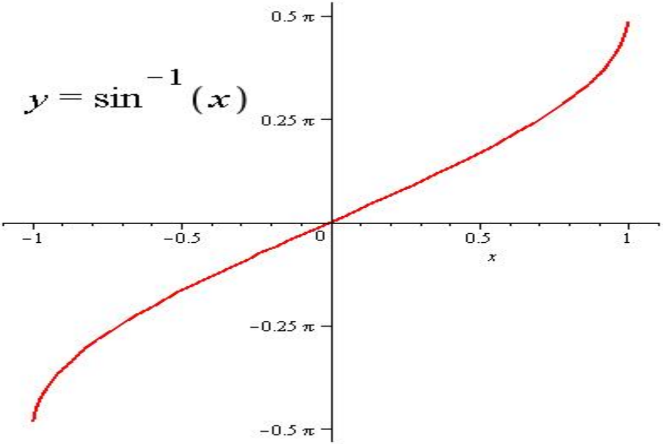


Figure: Note that the domain is $-1 \leq x \leq 1$ and the range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Function/Inverse Function Relationship

For every x in the interval $[-1, 1]$

$$\sin\left(\sin^{-1}(x)\right) = x$$

For every x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\sin^{-1}\left(\sin(x)\right) = x$$

Remark 1: If $x > 1$ or $x < -1$, the expression $\sin^{-1}(x)$ is not defined.

Remark 2: If $x > \frac{\pi}{2}$ or $x < -\frac{\pi}{2}$, the expression $\sin^{-1}\left(\sin(x)\right)$ IS defined, but IS NOT equal to x.

Example

Evaluate each expression if possible. If it is not defined, give a reason.

(a) $\sin^{-1}\left[\sin\left(\frac{\pi}{8}\right)\right]$ $= \frac{\pi}{8}$

$-\frac{\pi}{2} \le \frac{\pi}{8} \le \frac{\pi}{2}$

(b) $\sin^{-1}\left[\sin\left(\frac{4\pi}{3}\right)\right]$

$\frac{4\pi}{3} > \frac{\pi}{2}$

$\frac{4\pi}{3}$ is quad III

$= \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

Question

Recall that $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\sin\frac{5\pi}{6} = \frac{1}{2}$. And the range of the arcsine function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$\sin^{-1}\left[\sin\frac{5\pi}{6}\right] = \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{6}$$

(a) $\dfrac{5\pi}{6}$

(b) $\dfrac{\pi}{6}$

(c) $-\dfrac{\pi}{6}$

(d) $\dfrac{1}{2}$

(e) $-\dfrac{1}{2}$

The Inverse Cosine Function

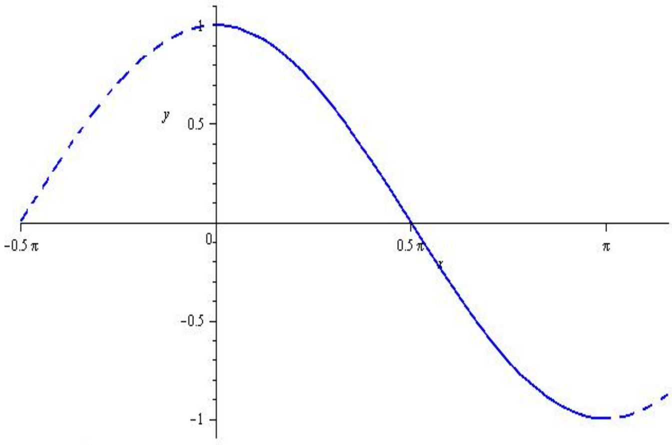


Figure: To define an inverse cosine function, we start by restricting the domain of $\cos(x)$ to the interval $[0, \pi]$

The Inverse Cosine Function (a.k.a. arccosine function)

Definition: For x in the interval $[-1, 1]$ the inverse cosine of x is denoted by either

$$\cos^{-1}(x) \quad \text{or} \quad \arccos(x)$$

Quads I and II

and is defined by the relationship

$$y = \cos^{-1}(x) \quad \Longleftrightarrow \quad x = \cos(y) \quad \text{where} \quad 0 \le y \le \pi.$$

The Domain of the Inverse Cosine is $-1 \le x \le 1$.

The Range of the Inverse Cosine is $0 \le y \le \pi$.

The Graph of the Arccosine

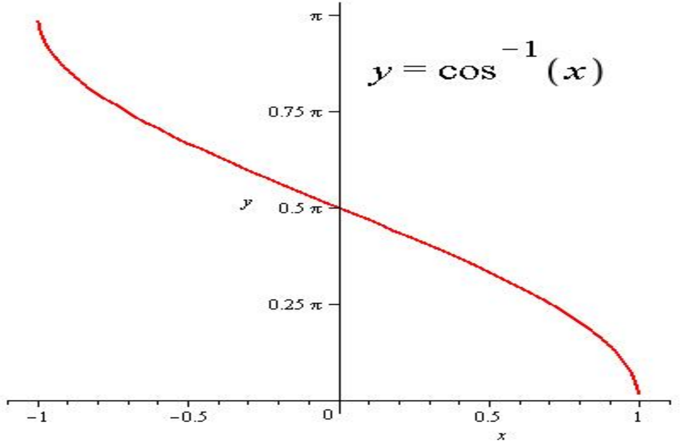


Figure: Note that the domain is $-1 \leq x \leq 1$ and the range is $0 \leq y \leq \pi$.

Function/Inverse Function Relationship

For every x in the interval $[-1, 1]$

$$\cos\left(\cos^{-1}(x)\right) = x$$

For every x in the interval $[0, \pi]$

$$\cos^{-1}\left(\cos(x)\right) = x$$

Remark 1: If $x > 1$ or $x < -1$, the expression $\cos^{-1}(x)$ is not defined.

Remark 2: If $x > \pi$ or $x < 0$, the expression $\cos^{-1}\left(\cos(x)\right)$ IS defined, but IS NOT equal to x.

Conceptual Definition

We can think of the inverse cosine function in the following way:

$\cos^{-1}(x)$ **is the *angle* between** 0 **and** π **whose cosine is** x**.**

Examples

Evaluate each expression exactly.

(a) $\cos^{-1}(0) = \frac{\pi}{2}$ because $\cos\frac{\pi}{2} = 0$ and $0 \le \frac{\pi}{2} \le \pi$

(b) $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ because $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $0 \le \frac{2\pi}{3} \le \pi$

Question

Recall $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

The exact value of $\quad \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) =$

(Remember the range is $[0, \pi]$.)

(a) $\frac{\pi}{4}$

(b) $-\frac{\pi}{4}$

(c) $\frac{3\pi}{4}$

(d) $-\frac{3\pi}{4}$

Evaluate the expression exactly

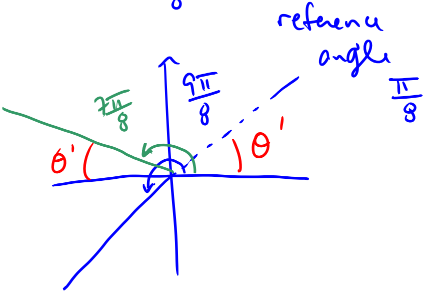
$$\cos^{-1}\left[\cos\left(\frac{9\pi}{8}\right)\right] = \frac{7\pi}{8}$$

Range for arccosine is $[0, \pi]$

and $\frac{9\pi}{8} > \pi$

This is the quadrant II angle whose reference angle is $\frac{\pi}{8}$

$$\cos\left(\frac{7\pi}{8}\right) = \cos\left(\frac{9\pi}{8}\right)$$



The Inverse Tangent Function

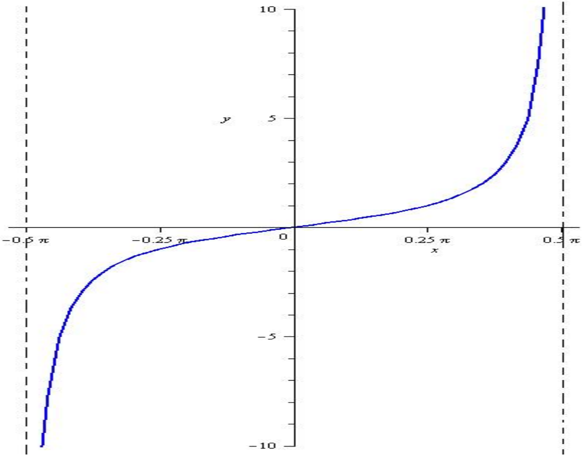


Figure: To define an inverse tangent function, we start by restricting the domain of $\tan(x)$ to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. **(Note the end points are NOT included!)**

The Inverse Tangent Function (a.k.a. arctangent function)

Definition: For all real numbers x, the inverse tangent of x is denoted by

$$\tan^{-1}(x) \quad \text{or by} \quad \arctan(x)$$

and is defined by the relationship

Quads I and IV

$$y = \tan^{-1}(x) \iff x = \tan(y) \quad \text{where} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

The Domain of the Inverse Tangent is $-\infty < x < \infty$.

The Range of the Inverse Cosine is $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (Note the strict inequalities.).

The Graph of the Arctangent

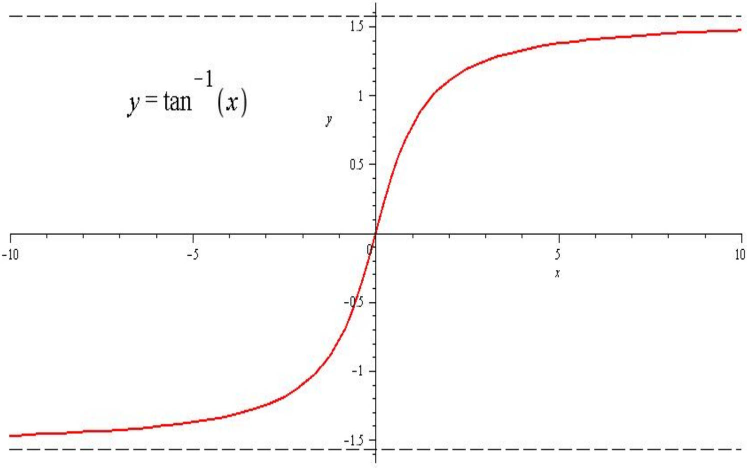


Figure: The domain is all real numbers and the range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$. The graph has two horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.

Function/Inverse Function Relationship

For all real numbers x

$$\tan\left(\tan^{-1}(x)\right) = x$$

For every x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\tan^{-1}\left(\tan(x)\right) = x$$

Remark 1: The expression $\tan^{-1}(x)$ is always well defined.

Remark 2: If $x > \frac{\pi}{2}$ or $x < -\frac{\pi}{2}$, the expression $\tan^{-1}\left(\tan(x)\right)$ MAY BE defined, but IS NOT equal to x.

Conceptual Definition

We can think of the inverse tangent function in the following way:

$\tan^{-1}(x)$ **is the *angle* between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x.**

Examples

Evaluate each expression exactly.

(a) $\tan^{-1}(-1)$ $= -\frac{\pi}{4}$ since $\tan\left(-\frac{\pi}{4}\right) = -1$

and $-\frac{\pi}{2} < -\frac{\pi}{4} < \frac{\pi}{2}$

(b) $\tan^{-1}(0)$ $= 0$

$\tan(0) = 0$ and

$-\frac{\pi}{2} < 0 < \frac{\pi}{2}$

Combining Functions

Evaluate each expression exactly if possible. If it is undefined, provide a reason.

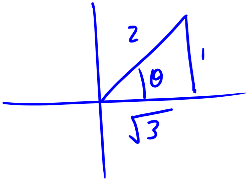
(a) $\tan\left[\arcsin\left(\frac{1}{2}\right)\right]$

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

If $\theta = \sin^{-1}\left(\frac{1}{2}\right)$ then

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$= \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$



(b) $\sin\left[\tan^{-1}\left(\frac{1}{4}\right)\right]$

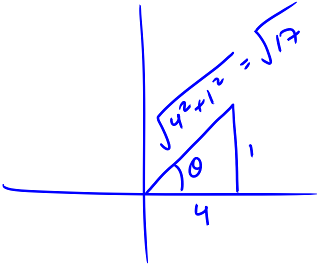
Let $\theta = \tan^{-1}\left(\frac{1}{4}\right)$ so

$\tan\theta = \frac{1}{4}$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

positive $\Rightarrow$ quadrant I

$= \frac{1}{\sqrt{17}}$

from the diagram



Algebraic Expressions

Write the expression purely algebraically (with no trigonometric functions).

$\tan\left[\sin^{-1}(x)\right]$

Let $\theta = \sin^{-1}(x)$ so $\sin\theta = x$

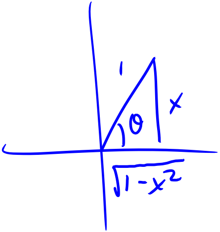
and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

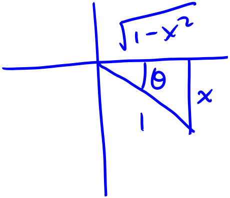
$= \dfrac{x}{\sqrt{1-x^2}}$

from the diagram

Quad I case

Quad IV case





Recap: Inverse Sine, Cosine, and Tangent

Function	Domain	Range	
$\sin^{-1}(x)$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	← quad I + IV
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$	← quad I and II
$\tan^{-1}(x)$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	← quad I and IV

Three Other Inverse Trigonometric Functions

There is disagreement about how to define the ranges of the inverse cotangent, cosecant, and secant functions!

The inverse Cotangent function is **typically** defined for all real numbers x by

$$y = \cot^{-1}(x) \quad \Longleftrightarrow \quad x = \cot(y) \quad \text{for} \quad 0 < y < \pi.$$

There is less consensus regarding the inverse secant and cosecant functions.

A Work-Around

We can use the fact that $\sec\theta = \frac{1}{\cos\theta}$ to find an expression for $\sec^{-1}(x)$. Suppose that

$$y = \sec^{-1}(x) \quad \text{and} \quad x = \sec(y).$$

Then

$$\frac{1}{x} = \cos(y) \quad \text{so that} \quad y = \cos^{-1}\left(\frac{1}{x}\right).$$

That is

$$\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right).$$

Use your result to compute (*an acceptable value for*) $\sec^{-1}(\sqrt{2})$

$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ so $\sec^{-1}(\sqrt{2}) = \frac{\pi}{4}$

A Work-Around

We can use the following compromise to compute inverse cotangent, secant, and cosecant values[2]:

For those values of x for which each side of the equation is defined

$$\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right),$$

$$\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right),$$

$$\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right).$$

[2] **Keep in mind that there is disagreement about the ranges!**

Question

True/False: Since $\cot\theta = \dfrac{1}{\tan\theta}$, it must be that $\cot^{-1} x = \dfrac{1}{\tan^{-1} x}$.

(a) True, and I'm confident.

(b) True, but I'm not confident.

(c) False, and I'm confident.

(d) False, but I'm not confident.

$\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right)$

but

$\cot^{-1}(x) \neq \dfrac{1}{\tan^{-1} x}$