Nov. 14 Math 1190 sec. 51 Fall 2016

Section 5.2: The Definite Integral

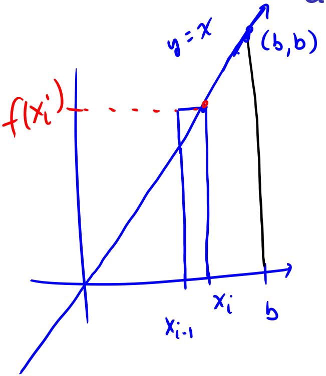
Recall that we gave the notation and the definition for the **definite integral of f from a to b**

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(c_i)\Delta x_i$$

where the limit is taken over all possible partitions of $[a,b]$.

- If the limit exists, it is a number. And we say that f is *integrable* on $[a,b]$.
- We use the phrase *integrating f with respect to x*.

Show that $\int_0^b x\,dx = \frac{b^2}{2}$ by using (i) a Riemann sum* and (ii) geometry.



(i) Form an equally spaced partition w/ n subintervals

$\Delta x = \frac{b-0}{n} = \frac{b}{n}$

$x_0 = 0,\ x_1 = \Delta x,\ x_2 = 2\Delta x, \ldots$

$x_i = i\Delta x = i\frac{b}{n}$

ith rectangle

height $= f(x_i) = x_i = i\frac{b}{n}$

width $= \Delta x = \frac{b}{n}$

*The following identity is useful

$$\sum_{i=1}^{n} i = \frac{n^2+n}{2},$$

Area of the ith rectangle is

$$\text{height} \times \text{width} = f(x_i)\,\Delta x = i\frac{b}{n}\cdot\frac{b}{n} = i\left(\frac{b}{n}\right)^2$$

Total area

$$A \approx \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} i\left(\frac{b}{n}\right)^2$$

$$= \left(\frac{b}{n}\right)^2 \sum_{i=1}^{n} i = \frac{b^2}{n^2}\left(\frac{n^2+n}{2}\right)$$

$$= \frac{b^2}{2}\,\frac{n^2+n}{n^2} = \frac{b^2}{2}\left(\frac{n^2}{n^2}+\frac{n}{n^2}\right) = \frac{b^2}{2}\left(1+\frac{1}{n}\right)$$

$$A \approx \frac{b^2}{2}\left(1+\frac{1}{n}\right)$$

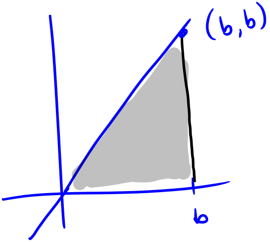
The true area is obtained by taking $n \to \infty$.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2}(1+0) = \frac{b^2}{2}$$

This shows that $\displaystyle\int_0^b x\,dx = \frac{b^2}{2}$

(ii) Using geometry, the region is a triangle



base $B = b$

height $H = b$

$$\text{Area} = \frac{1}{2} B \cdot H = \frac{1}{2} b \cdot b = \frac{b^2}{2}$$

again, $\int_0^b x \, dx = \frac{b^2}{2}$

Section 5.3: The Fundamental Theorem of Calculus

Suppose f is continuous on the interval $[a, b]$. For $a \leq x \leq b$ define a new function

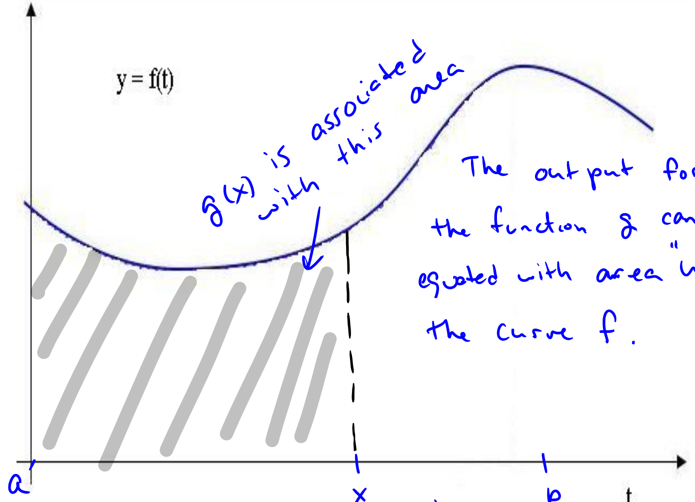
$$g(x) = \int_a^x f(t)\,dt$$

How can we understand this function, and what can be said about it?

t is a dummy variable of integration

x is an independent variable

Geometric interpretation of $g(x) = \int_a^x f(t)\,dt$



The output for the function g can be equated with area "under" the curve f.

Figure

pick x between a and b

Theorem: The Fundamental Theorem of Calculus (part 1)

If f is continuous on $[a, b]$ and the function g is defined by

$$g(x) = \int_a^x f(t)\, dt \quad \text{for} \quad a \le x \le b,$$

then g is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$g'(x) = f(x).$$

This means that the new function g is an **antiderivative** of f on (a, b)!
"FTC" = "fundamental theorem of calculus"

This can be written as $\dfrac{d}{dx} \displaystyle\int_a^x f(t)\, dt = f(x)$

Example:

Evaluate each derivative.

(a) $\displaystyle \frac{d}{dx}\int_0^x \sin^2(t)\,dt$ $=$ $\sin^2(x)$

here $f(t) = \sin^2 t$

so $f(x) = \sin^2 x$

$a = 0$

(b) $\displaystyle \frac{d}{dx}\int_4^x \frac{t-\cos t}{t^4+1}\,dt$ $=$ $\displaystyle \frac{x-\cos x}{x^4+1}$

here $f(t) = \dfrac{t-\cos t}{t^4+1}$

so $f(x) = \dfrac{x-\cos x}{x^4+1}$

$a = 4$

Question

Evaluate $\dfrac{d}{dx}\displaystyle\int_2^x e^{3t^2}\,dt$

(a) e^{3x^2}

$f(t) = e^{3t^2}$ so $f(x) = e^{3x^2}$

(b) $6xe^{3x^2}$

(c) $e^{3x^2} - e^{12}$