Nov. 14 Math 1190 sec. 52 Fall 2016

Section 5.2: The Definite Integral

Recall that we gave the notation and the definition for the **definite integral of f from a to b**

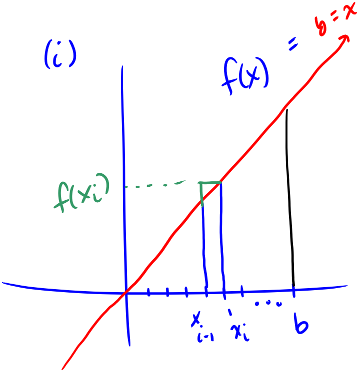
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where the limit is taken over all possible partitions of $[a, b]$.

- If the limit exists, it is a number. And we say that f is *integrable* on $[a, b]$.
- We use the phrase *integrating f with respect to x*.

Show that $\int_0^b x\,dx = \frac{b^2}{2}$ by using (i) a Riemann sum* and (ii) geometry.

(i)



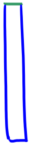
Let's use an equally space partition w/ n subintervals

$$\Delta x = \frac{b-a}{n} = \frac{b-0}{n} = \frac{b}{n}$$

$$x_0 = 0, \quad x_1 = \Delta x = \frac{b}{n}, \quad x_2 = 2\Delta x = 2\cdot\frac{b}{n}$$

$$\Rightarrow \quad x_i = 0 + i\Delta x = i\frac{b}{n}$$

i^{th} rectangle



height $= f(x_i) = x_i = i\frac{b}{n}$

width $= \Delta x = \frac{b}{n}$

*The following identity is useful

$$\sum_{i=1}^{n} i = \frac{n^2+n}{2},$$

The area of the ith rectangle is

$$\text{height} \times \text{width} = f(x_i)\,\Delta x = i\,\frac{b}{n} \cdot \frac{b}{n} = i\left(\frac{b}{n}\right)^2$$

$$\int_0^b x\,dx \approx \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} i\left(\frac{b}{n}\right)^2$$ the sum of the rectangles.

To get the true value, we need to take $n \to \infty$.

Let's simplify first

$$\sum_{i=1}^{n} i\left(\frac{b}{n}\right)^2 = \left(\frac{b}{n}\right)^2 \sum_{i=1}^{n} i = \left(\frac{b}{n}\right)^2 \left(\frac{n^2+n}{2}\right)$$

$$= \frac{b^2}{n^2}\left(\frac{n^2+n}{2}\right) = \frac{b^2}{2}\left(\frac{n^2+n}{n^2}\right)$$

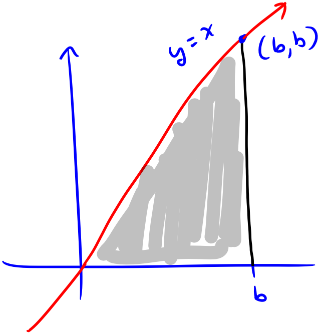
$$= \frac{b^2}{2}\left(\frac{n^2}{n^2} + \frac{n}{n^2}\right) = \frac{b^2}{2}\left(1 + \frac{1}{n}\right)$$

$$\int_0^b x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} i\left(\frac{b}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{b^2}{2}\left(1 + \frac{1}{n}\right) = \frac{b^2}{2}(1+0) = \frac{b^2}{2}$$

(ii) Using geometry



$$\int_0^b x\,dx = \text{area in the triangle}$$

Base $B = b$

Height $H = b$

$$\text{area} = \frac{1}{2} BH = \frac{1}{2} \cdot b \cdot b = \frac{b^2}{2}$$

So again $\int_0^b x\,dx = \frac{b^2}{2}$

Section 5.3: The Fundamental Theorem of Calculus

Suppose f is continuous on the interval $[a, b]$. For $a \leq x \leq b$ define a new function

$$g(x) = \int_a^x f(t)\, dt$$

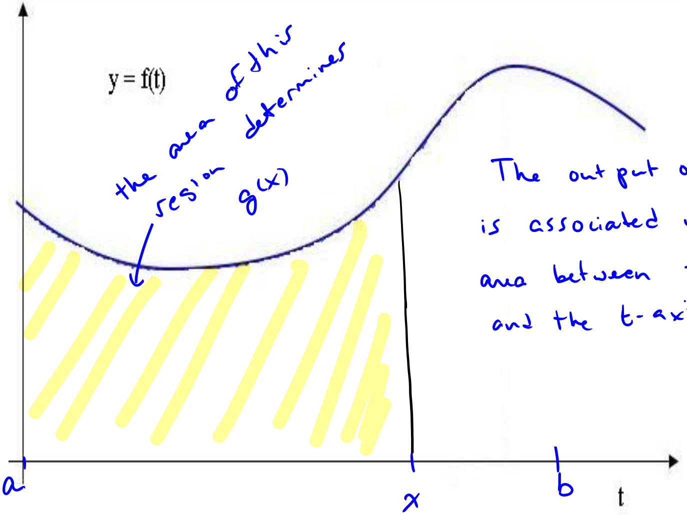
How can we understand this function, and what can be said about it?

t is the dummy variable of integration

x is the independent variable

g(x) the output for input x

Geometric interpretation of $g(x) = \int_a^x f(t)\,dt$



The output of g is associated with area between $f(t)$ and the t-axis.

Figure

pick an x in [a,b]

Theorem: The Fundamental Theorem of Calculus (part 1)

If f is continuous on $[a, b]$ and the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{for} \quad a \leq x \leq b,$$

then g is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$g'(x) = f(x).$$

This means that the new function g is an **antiderivative** of f on (a, b)!
"FTC" = "fundamental theorem of calculus"

$$g'(x) = f(x) \Rightarrow \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Example:

Evaluate each derivative.

(a) $\dfrac{d}{dx}\displaystyle\int_0^x \sin^2(t)\,dt$ $= \sin^2(x)$

here $f(t) = \sin^2(t)$

so $f(x) = \sin^2(x)$

$a = 0$

(b) $\dfrac{d}{dx}\displaystyle\int_4^x \dfrac{t-\cos t}{t^4+1}\,dt$ $= \dfrac{x-\cos x}{x^4+1}$

here $f(t) = \dfrac{t-\cos t}{t^4+1}$

so $f(x) = \dfrac{x-\cos x}{x^4+1}$

$a = 4$

Question

Evaluate $\dfrac{d}{dx}\displaystyle\int_{2}^{x} e^{3t^2}\,dt$

(a) e^{3x^2}

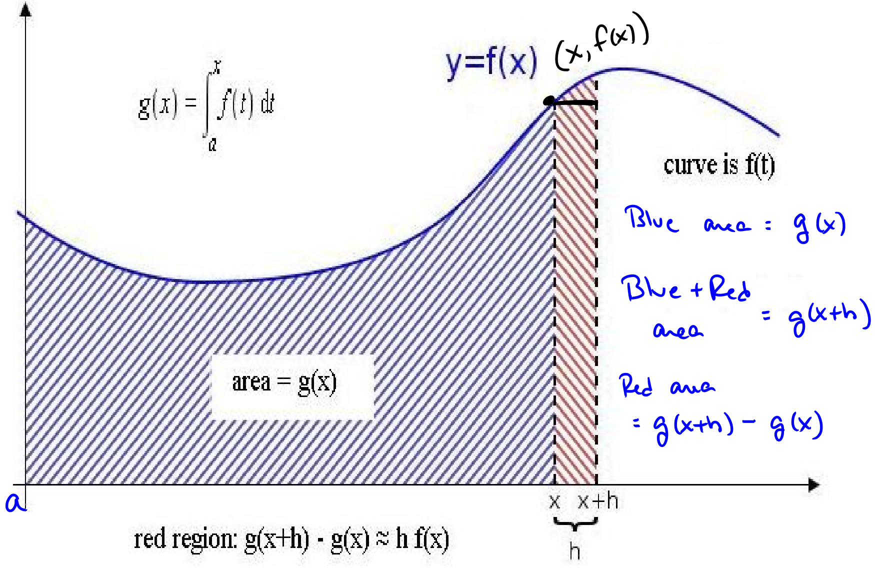
$f(t) = e^{3t^2}$

so $f(x) = e^{3x^2}$

(b) $6xe^{3x^2}$

(c) $e^{3x^2} - e^{12}$

Geometric Argument of FTC



By definition:

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$g(x+h) - g(x) \approx$ area of red rectangle

$= \text{height} \times \text{width} = f(x) \cdot h$

We have

$$g(x+h) - g(x) \approx h f(x)$$

$$\Rightarrow \quad \frac{g(x+h) - g(x)}{h} \approx f(x)$$

The smaller h is, the better our approximation.

We'll take $h \to 0$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \approx \lim_{h \to 0} f(x)$$

$$= f(x)$$

since the approximation becomes exact as $h \to 0$.

Chain Rule with FTC

Evaluate each derivative.

(a) $\dfrac{d}{dx}\displaystyle\int_0^{x^2} t^3\,dt$

Chain rule: $\dfrac{d}{dx}F(u) = F'(u)\cdot\dfrac{du}{dx}$

if $u = u(x)$

This is a composition w/ outside function $F(u) = \displaystyle\int_0^u t^3\,dt$, here $u = x^2$

by the FTC $\rightarrow$ $F'(u) = u^3$ power rule $\rightarrow$ $\dfrac{du}{dx} = 2x$

$$\frac{d}{dx}\int_0^{x^2} t^3\,dt = F'(u)\frac{du}{dx} = u^3\cdot(2x) = (x^2)^3\cdot(2x) = 2x^7$$

(b) $\dfrac{d}{dx}\displaystyle\int_x^7 \cos(t^2)\,dt$

Recall $\displaystyle\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$

$$= \frac{d}{dx}\left(-\int_7^x \cos(t^2)\,dt\right)$$

$$= -\frac{d}{dx}\int_7^x \cos(t^2)\,dt = -\cos(x^2)$$

Question

(c) $\dfrac{d}{dx}\displaystyle\int_3^{\sin x} \frac{1}{1+t^3}\,dt$

$$F(u) = \int_3^u \frac{1}{1+t^3}\,dt$$

and $u = \sin x$

$$F'(u) = \frac{1}{1+u^3} = \frac{1}{1+\sin^3 x}$$

$$\frac{du}{dx} = \cos x$$

$$\Rightarrow F'(u)\frac{du}{dx} = \frac{1}{1+\sin^3 x}\cdot \cos x = \frac{\cos x}{1+\sin^3 x}$$

(a) $\frac{1}{1+\sin^3 x}$

(b) $\frac{\cos x}{1+\sin^3 x}$

(c) $\frac{\cos x}{1+x^3}$

(d) $\frac{-3\sin^2 x}{(1+\sin^3 x)^2}$