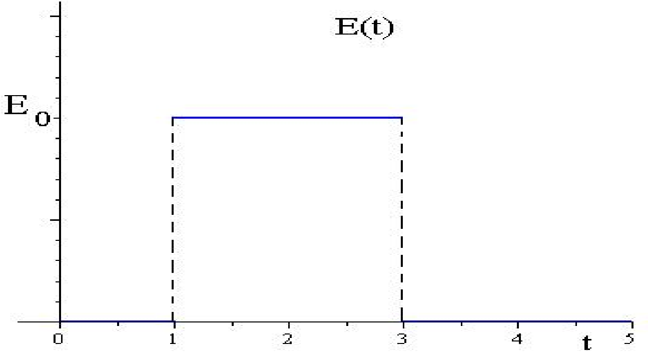
November 14 Math 2306 sec. 53 Fall 2018

Section 16: Laplace Transforms of Derivatives and IVPs

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.



LR Circuit Example

We found that we could express E as

$$E(t) = E_0\mathscr{U}(t-1) - E_0\mathscr{U}(t-3)$$

so that the IVP we wish to solve is

$$\frac{di}{dt} + 10i = E_0\mathscr{U}(t-1) - E_0\mathscr{U}(t-3), \quad i(0) = 0.$$

$$\mathscr{L}\{i(t)\} = I(s)$$

$$\mathscr{L}\{i' + 10i\} = \mathscr{L}\{E_0\mathcal{U}(t-1) - E_0\mathcal{U}(t-3)\}$$

$$\mathscr{L}\{i'\} + 10\mathscr{L}\{i\} = E_0\mathscr{L}\{\mathcal{U}(t-1)\} - E_0\mathscr{L}\{\mathcal{U}(t-3)\}$$

$$sI(s) - \underset{\overset{\parallel}{0}}{i(0)} + 10I(s) = E_0\frac{e^{-s}}{s} - E_0\frac{e^{-3s}}{s}$$

$$(s+10)I(s) = \frac{E_0 e^{-s}}{s} - \frac{E_0 e^{-3s}}{s}$$

$$I(s) = \frac{E_0 e^{-s}}{s(s+10)} - \frac{E_0 e^{-3s}}{s(s+10)}$$

We'll do a partial fraction decomp on $\frac{1}{s(s+10)}$

$$\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

$$1 = A(s+10) + Bs$$

$s=0 \qquad 1 = 10A \implies A = \frac{1}{10}$

$s=-10 \qquad 1 = -10B \implies B = \frac{-1}{10}$

So

$$I(s) = \frac{E_0}{10}e^{-s}\left(\frac{1}{s} - \frac{1}{s+10}\right) - \frac{E_0}{10}e^{-3s}\left(\frac{1}{s} - \frac{1}{s+10}\right)$$

we'll use

$$\mathscr{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)\mathscr{U}(t-a)$$

where $f(t) = \mathscr{L}^{-1}\left\{F(s)\right\}$

We need

$$\mathscr{L}^{-1}\left\{\frac{E_0}{10}\left(\frac{1}{s} - \frac{1}{s+10}\right)\right\}$$

$$= \frac{E_0}{10}\left(\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\}\right)$$

$$= \frac{E_0}{10}\left(1 - e^{-10t}\right)$$

$$i(t) = \mathcal{L}^{-1}\left\{I(s)\right\} = \mathcal{L}^{-1}\left\{\frac{E_0}{10}e^{-s}\left(\frac{1}{s} - \frac{1}{s+10}\right) - \frac{E_0}{10}e^{-3s}\left(\frac{1}{s} - \frac{1}{s+10}\right)\right\}$$

$$\boxed{i(t) = \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right)\mathcal{U}(t-1) - \frac{E_0}{10}\left(1 - e^{-10(t-3)}\right)\mathcal{U}(t-3)}$$

Let's write this in "stacked form":

$$u(t-1) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases} \qquad u(t-3) = \begin{cases} 0, & 0 \le t < 3 \\ 1, & t \ge 3 \end{cases}$$

For $0 \le t < 1$ $\quad u(t-1) = 0$ and $u(t-3) = 0$

$$i(t) = 0$$

For $1 \le t < 3$ $\quad u(t-1) = 1$ and $u(t-3) = 0$

$$i(t) = \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right)$$

For $t \geq 3$, $u(t-1) = 1$ $\quad u(t-3) = 1$

$$i(t) = \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right) - \frac{E_0}{10}\left(1 - e^{-10(t-3)}\right)$$

$$= \frac{E_0}{10}\left(e^{-10(t-3)} - e^{-10(t-1)}\right)$$

$$i(t) = \begin{cases} 0, & 0 \leq t < 1 \\ \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right), & 1 \leq t < 3 \\ \frac{E_0}{10}\left(e^{-10(t-3)} - e^{-10(t-1)}\right), & t \geq 3 \end{cases}$$

Section 17: Fourier Series: Trigonometric Series

Some Preliminary Concepts

Suppose two functions f and g are integrable on the interval $[a, b]$. We define the **inner product** of f and g on $[a, b]$ as

$$< f, g >= \int_a^b f(x)g(x)\,dx.$$

We say that f and g are **orthogonal** on $[a, b]$ if

$$< f, g >= 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

Properties of an Inner Product

Let f, g, and h be integrable functions on the appropriate interval and let c be any real number. The following hold

(i) $< f, g >=< g, f >$

(ii) $< f, g + h >=< f, g > + < f, h >$

(iii) $< cf, g >= c < f, g >$

(iv) $< f, f >\geq 0$ and $< f, f >= 0$ if and only if $f = 0$

Orthogonal Set

A set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$< \phi_m, \phi_n >= \int_a^b \phi_m(x)\phi_n(x)\,dx = 0 \quad \text{whenever} \quad m \neq n.$$

Note that any function $\phi(x)$ that is not identically zero will satisfy

$$< \phi, \phi >= \int_a^b \phi^2(x)\,dx > 0.$$

Hence we define the **square norm** of ϕ (on $[a, b]$) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x)\,dx}.$$

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

Evaluate $\langle \cos(nx), 1 \rangle$ and $\langle \sin(mx), 1 \rangle$. $n, m \geq 1$

$a = -\pi$
$b = \pi$

$$\langle \cos(nx), 1 \rangle = \int_{-\pi}^{\pi} \cos(nx) \cdot 1 \, dx$$

$$= \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi} = \frac{1}{n} \sin(n\pi) - \frac{1}{n} \sin(-n\pi) = 0$$

$$\langle \sin(mx), 1 \rangle = \int_{-\pi}^{\pi} \sin(mx) \cdot 1 \, dx = \frac{-1}{m} \cos(mx) \Big|_{-\pi}^{\pi}$$

$\cos(-\theta) = \cos\theta$

$$= \frac{-1}{m} \cos(m\pi) - \frac{-1}{m} \cos(-m\pi)$$

$$= \frac{-1}{m} \cos(m\pi) + \frac{1}{m} \cos(m\pi) = 0$$

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

It can easily be verified that

$$\int_{-\pi}^{\pi} \cos nx\, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx\, dx = 0 \quad \text{for all} \quad n, m \geq 1,$$

$$\int_{-\pi}^{\pi} \cos nx\, \sin mx\, dx = 0 \quad \text{for all} \quad m, n \geq 1, \quad \text{and}$$

$$\int_{-\pi}^{\pi} \cos nx\, \cos mx\, dx = \int_{-\pi}^{\pi} \sin nx\, \sin mx\, dx = \left\{ \begin{array}{ll} 0, & m \neq n \\ \pi, & n = m \end{array} \right. ,$$

An Orthogonal Set of Functions on $[-\pi, \pi]$

These integral values indicated that the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$$

is an orthogonal set on the interval $[-\pi, \pi]$.

An Orthogonal Set of Functions on $[-p, p]$

This set can be generalized by using a simple change of variables $t = \frac{\pi x}{p}$ to obtain the orthogonal set on $[-p, p]$

$$\left\{1, \cos\frac{n\pi x}{p}, \sin\frac{m\pi x}{p} \,|\, n, m \in \mathbb{N}\right\}$$

There are many interesting and useful orthogonal sets of functions (on appropriate intervals). What follows is readily extended to other such (infinite) sets.

Fourier Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines.**

Task: Find coefficients (numbers) a_0, $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that[1]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

[1] We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) .$$

The question of convergence naturally arises when we wish to work with infinite series. To highlight convergence considerations, some authors prefer not to use the equal sign when expressing a Fourier series and instead write

$$f(x) \sim \frac{a_0}{2} + \cdots$$

Herein, we'll use the equal sign with the understanding that equality may not hold at each point.

Convergence will be address later.

Finding an Example Coefficient

For a known function f defined on $(-\pi, \pi)$, assume there is such a series[2]. Let's find the coefficient b_4.

Multiply by $\sin(4x)$

$$f(x)\sin(4x) = \frac{a_0}{2}\sin(4x) + \sum_{n=1}^{\infty}\left(a_n \cos nx \sin(4x) + b_n \sin nx \sin(4x)\right).$$

Integrate from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = \underbrace{\int_{-\pi}^{\pi} \frac{a_0}{2}\sin(4x)\,dx}_{= 0}$$

[2]We will also assume that the order of integrating and summing can be interchanged.

$$+ \sum_{n=1}^{\infty} \left(\underbrace{\int_{-\pi}^{\pi} a_n \cos(nx) \sin(4x)\,dx}_{=\,0} + \int_{-\pi}^{\pi} b_n \sin(nx) \sin(4x)\,dx \right)$$

$$\int_{-\pi}^{\pi} f(x) \sin(4x)\,dx = \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi} \sin(nx) \sin(4x)\,dx}_{=\begin{cases} 0, & n \neq 4 \\ \pi, & n = 4 \end{cases}}$$

$$= b_4 \pi$$

$$b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx$$