November 14 Math 2306 sec. 56 Fall 2017

Section 17: Fourier Series: Trigonometric Series

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

Where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}$$

Example

Find the Fourier series of $f(x) = e^x$ for $-\pi < x < \pi$. The following results are useful

$$\int e^x \cos(nx)\,dx = \frac{e^x}{n^2+1}\cos(nx) + \frac{ne^x}{n^2+1}\sin(nx)$$

$$\int e^x \sin(nx)\,dx = \frac{e^x}{n^2+1}\sin(nx) - \frac{ne^x}{n^2+1}\cos(nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{n^2+1}\cos(nx) + \frac{ne^x}{n^2+1}\sin(nx)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{n^2+1}\underbrace{\cos(n\pi)}_{=(-1)^n} - \frac{e^{-\pi}}{n^2+1}\cos(-n\pi) + \frac{ne^{\pi}}{n^2+1}\underbrace{\cancel{\sin(n\pi)}}_{0} - \frac{ne^{-\pi}}{n^2+1}\underbrace{\cancel{\sin(-n\pi)}}_{0}\right]$$

$$= \frac{(-1)^n}{\pi(n^2+1)}\left(e^{\pi} - e^{-\pi}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{n^2+1}\sin(nx) - \frac{ne^x}{n^2+1}\cos(nx)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{n^2+1}\underbrace{\sin(n\pi)}_{0} - \frac{e^{-\pi}}{n^2+1}\underbrace{\sin(-n\pi)}_{0} - \left(\frac{n e^{\pi}}{n^2+1}\cos(n\pi) - \frac{n e^{-\pi}}{n^2+1}\cos(-n\pi) \right) \right]$$

$$= \frac{-n(-1)^n}{\pi(n^2+1)}\left(e^{\pi} - e^{-\pi}\right)$$

$$= \frac{n(-1)^{n+1}}{\pi(n^2+1)}\left(e^{\pi} - e^{-\pi}\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x}\,dx = \frac{1}{\pi} e^{x}\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$f(x) = \frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right) + \sum_{n=1}^{\infty}\left(\frac{(-1)^n\left(e^{\pi}-e^{-\pi}\right)}{\pi(n^2+1)}\cos(nx) + \frac{n(-1)^{n+1}\left(e^{\pi}-e^{-\pi}\right)}{\pi(n^2+1)}\sin(nx)\right)$$

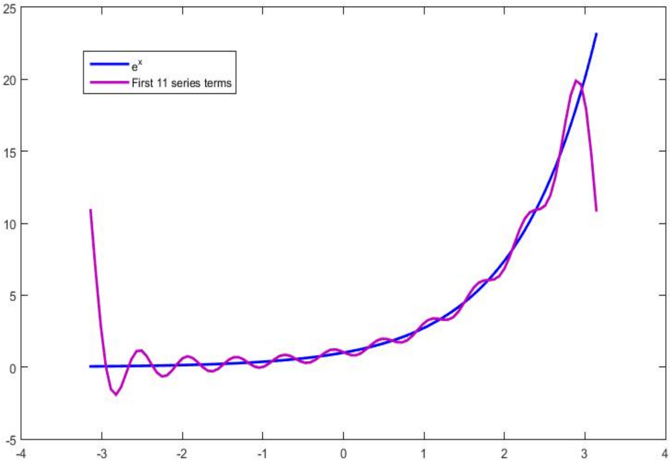


Figure: The function $f(x) = e^x$ on $(-\pi, \pi)$ together with the partial sum of the first 11 terms of the Fourier series.

Fourier Series on an interval $(-p, p)$

The set of functions $\left\{1, \cos\left(\frac{n\pi x}{p}\right), \sin\left(\frac{m\pi x}{p}\right) \middle| n, m \geq 1\right\}$ is orthogonal on $[-p, p]$. Moreover, we have the properties

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\, dx = 0 \quad \text{and} \quad \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \ \text{ for all } \ n, m \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \quad \text{for all} \quad m, n \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \cos\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases},$$

$$\int_{-p}^{p} \sin\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases}.$$

Fourier Series on an interval $(-p, p)$

The orthogonality relations provide for an expansion of a function f defined on $(-p, p)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right)$$

where

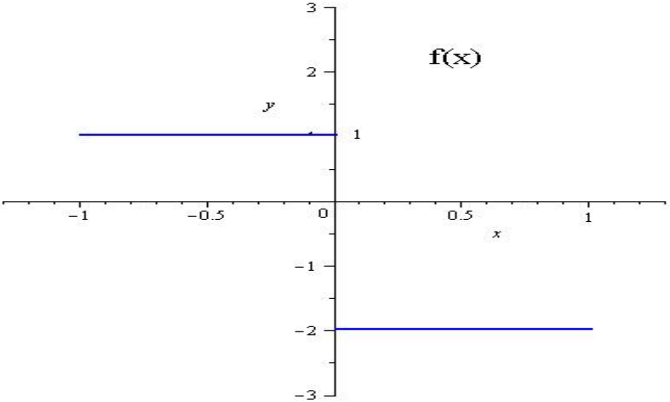
$$\begin{aligned}
a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx, \\
a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx, \quad \text{and} \\
b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx
\end{aligned}$$

Find the Fourier series of f

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}$$

f is given on $(-1,1)$ so $p=1$

for $p=1$ $\quad \frac{n\pi x}{p} = \frac{n\pi x}{1} = n\pi x$



$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases} \qquad p = 1$$

$$f(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} dx + \int_{0}^{1} (-2)\,dx$$

$$x\Big|_{-1}^{0} - 2x\Big|_{0}^{1} = (0-(-1)) - 2(1-0) = -1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0}\cos(n\pi x)\,dx + \int_{0}^{1}(-2)\cos(n\pi x)\,dx$$

$$\frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} - 2\,\frac{1}{n\pi}\sin(n\pi x)\Big|_{0}^{1}$$

$$\frac{1}{n\pi}\Big(\underbrace{\sin(0)}_{0} - \underbrace{\sin(-n\pi)}_{0}\Big) - \frac{2}{n\pi}\Big(\underbrace{\sin(n\pi)}_{0} - \underbrace{\sin(0)}_{0}\Big)$$

$a_n = 0$ for all $n \geq 1$.

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\,\sin(n\pi x)\,dx = \int_{-1}^{0}\sin(n\pi x)\,dx + \int_{0}^{1}(-2)\sin(n\pi x)\,dx$$

$$= \frac{-1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0} + \frac{2}{n\pi}\cos(n\pi x)\Big|_{0}^{1}$$

$$= \frac{-1}{n\pi}\Big(\underbrace{\cos(0)}_{=\,1} - \underbrace{\cos(-n\pi)}_{=\,(-1)^n}\Big) + \frac{2}{n\pi}\Big(\underbrace{\cos(n\pi)}_{=\,(-1)^n} - \underbrace{\cos(0)}_{=\,1}\Big)$$

$$= \frac{-1}{n\pi} + \frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n\pi} - \frac{2}{n\pi} = \frac{3}{n\pi}\left((-1)^n - 1\right)$$

$a_0 = -1, \quad a_n = 0, \quad b_n = \frac{3}{n\pi}\left((-1)^n - 1\right)$

So

$$f(x) = \frac{-1}{2} + \sum_{n=1}^{\infty} \frac{3}{n\pi}\left((-1)^n - 1\right)\sin(n\pi x)$$

Note when $x = 0$, the series is

$$\frac{-1}{2} + \sum_{n=1}^{\infty} \frac{3}{n\pi}\left((-1)^n - 1\right)\underbrace{\sin(0)}_{=0} = \frac{-1}{2}$$

← average value at the jump!

Convergence?

The last example gave the series

$$f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

This example raises an interesting question: The function f is not continuous on the interval $(-1, 1)$. However, each term in the Fourier series, and any partial sum thereof, is obviously continuous. This raises questions about properties (e.g. continuity) of the series. More to the point, we may ask: *what is the connection between f and its Fourier series at the point of discontinuity?*

This is the convergence issue mentioned earlier.

Convergence of the Series

Theorem: If f is continuous at x_0 in $(-p, p)$, then the series converges to $f(x_0)$ at that point. If f has a jump discontinuity at the point x_0 in $(-p, p)$, then the series **converges in the mean** to the average value

$$\frac{f(x_0-) + f(x_0+)}{2} \stackrel{\text{def}}{=} \frac{1}{2}\left(\lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x)\right)$$

at that point.

Periodic Extension:
The series is also defined for x outside of the original domain $(-p, p)$. The extension to all real numbers is $2p$-periodic.