November 14 Math 3260 sec. 57 Fall 2017

Section 5.3: Diagonalization

Determine the eigenvalues of the matrix D^3 where $D = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$.

Diagonal Matrices

Recall: A matrix D is diagonal if it is both upper and lower triangular (its only nonzero entries are on the diagonal).

Note: If D is diagonal with diagonal entries d_{ii}, then D^k is diagonal with diagonal entries d_{ii}^k for positive integer k. Moreover, the eigenvalues of D are the diagonal entries.

Powers and Similarity

Show that if A and B are similar, with similarity tranformation matrix P, then A^k and B^k are similar with the same matrix P.

Diagonalizability

Defintion: An $n \times n$ matrix A is called **diagonalizable** if it is similar to a diagonal matrix D. That is, provided there exists a nonsingular matrix P such that $D = P^{-1}AP$—i.e. $A = PDP^{-1}$.

Theorem: The $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case, the matrix P is the matrix whose columns are the n linearly independent eigenvectors of A.

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Theorem (a second on diagonalizability)

Recall: (sec. 5.1) If λ_1 and λ_2 are distinct eigenvalues of a matrix, the corresponding eigenvectors are linearly independent.

Theorem: If the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Note: This is a *sufficiency* condition, not a *necessity* condition. We've already seen a matrix with a repeated eigenvalue that was diagonalizable.

Theorem (a third on diagonalizability)

Theorem: Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

(a) The geometric multiplicity (dimension of the eigenspace) of λ_k is less than or equal to the algebraic multiplicity of λ_k.

(b) The matrix is diagonalizable if and only if the sum of the geometric multiplicities is n—i.e. the sum of dimensions of all eigenspaces is n so that there are n linearly independent eigenvectors.

(c) If A is diagonalizable, and $\mathcal{B}_k$ is a basis for the eigenspace for λ_k, then the collection (union) of bases $\mathcal{B}_1, \ldots, \mathcal{B}_p$ is a basis for $\mathbb{R}^n$.

Remark: The union of the bases referred to in part (c) is called an **eigenvector basis** for $\mathbb{R}^n$. (Of course, one would need to reference the specific matrix.)

Example

Diagonalize the matrix if possible. $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

Example Continued...

Find A^4 where $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

Section 5.4: Eigenvectors and Linear Transformations

Recall: A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ can always be written in terms of a matrix product $T(\mathbf{x}) = A\mathbf{x}$ where A is the standard matrix for T.

Questions: If A happens to be diagonalizable (A similar to D) is there a way to rewrite T in terms of D to take advantage of the nice diagonal matrix?

If we replace $\mathbb{R}^n$ and $\mathbb{R}^m$ with other (finite dimensional) vectors spaces (e.g. $\mathbb{P}_n$ or $M^{2\times 2}$) can we still write T in terms of a matrix?

The second question

Recall: If V is an n dimensional vector space with ordered basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then for each $\mathbf{x}$ in V we can consider its coordinate vector, which is a vector in $\mathbb{R}^n$, $[\mathbf{x}]_{\mathcal{B}}$.

Schematic: Let $T : V \longrightarrow W$ be linear with V and W vector spaces of dimension n and m, respectively. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for V, and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ be a basis for W. Then build a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ that captures the relationship

$$[\mathbf{x}]_{\mathcal{B}} \quad \mapsto \quad [T(\mathbf{x})]_{\mathcal{C}}\,.$$

This mapping has a matrix M.

Matrix of a Linear Transformation

Let $T : V \longrightarrow W$. For $\mathbf{x}$ in V suppose

$$[\mathbf{x}]_{\mathcal{B}} = (r_1, r_2, \ldots, r_n)$$

so that

$$T(\mathbf{x}) = r_1 T(\mathbf{b}_1) + r_2 T(\mathbf{b}_2) + \cdots + r_n T(\mathbf{b}_n).$$

Next, using the coordinate mapping on W

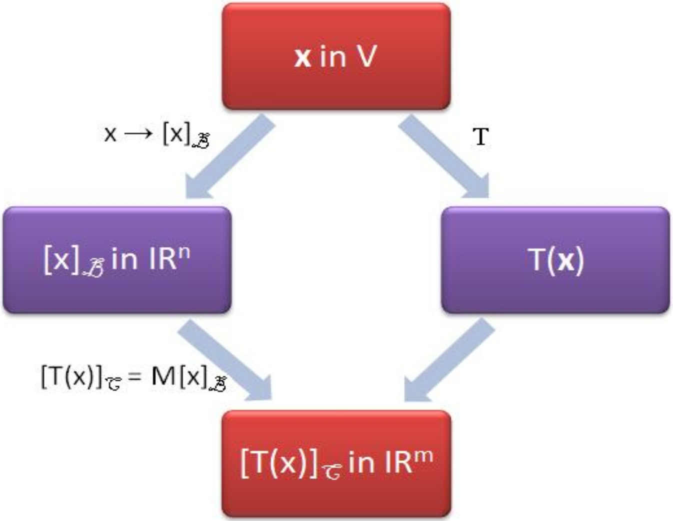
$$[T(\mathbf{x})]_{\mathcal{C}} = r_1[T(\mathbf{b}_1)]_{\mathcal{C}} + r_2[T(\mathbf{b}_2)]_{\mathcal{C}} + \cdots + r_n[T(\mathbf{b}_n)]_{\mathcal{C}}.$$

This is a vector equation in $\mathbb{R}^m$! So we can write it as a matrix product

$$[T(\mathbf{x})]_{\mathcal{C}} = M[\mathbf{x}]_{\mathcal{B}}$$

where the columns of M are the vectors $[T(\mathbf{b}_i)]_{\mathcal{C}}$ for $i = 1, \ldots, n$. M is called the **matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.**

Matrix of a Linear Transformation



Example

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be bases for V and W respectively. And suppose the linear transformation $T : V \longrightarrow W$ is defined by the properties

$$T(\mathbf{b}_1) = \mathbf{c}_1 - 3\mathbf{c}_2 + 4\mathbf{c}_3, \quad \text{and} \quad T(\mathbf{b}_2) = 5\mathbf{c}_2 - 2\mathbf{c}_3.$$

Find the matrix M for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$

An Example with $V = W$

Find the matrix for $T : \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ relative to the basis $\mathcal{B} = \{1, t, t^2\}$ where T is defined by

$$T(p_0 + p_1 t + p_2 t^2) = p_1 - 3p_1 t + (p_0 - p_2)t^2.$$

Use the results to find $T(2t^2 + t - 5)$.

Transformation from V to V

Definition: If V is an n dimensional vector space and $T : V \longrightarrow V$ is linear, and if a single basis $\mathcal{B}$ is used to construct a matrix for T, then this matrix is called **the matrix for T relative to $\mathcal{B}$** or simply **the $\mathcal{B}$ matrix of T.** It will be denoted by

$$[T]_{\mathcal{B}}.$$

Example $V = \mathbb{R}^2$

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$T(x_1, x_2) = (3x_1 - 2x_2, x_1 + 4x_2).$$

Find the $\mathcal{B}$ matrix of T for the basis[1]

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\}.$$

[1] **Remember that $[\mathbf{x}] = P_{\mathcal{B}}^{-1}\mathbf{x}$ where $P_{\mathcal{B}} = [\mathbf{b}_1\ \mathbf{b}_2]$.**

Example Continued...

Find the standard matrix A for $T(x_1, x_2) = (3x_1 - 2x_2, x_1 + 4x_2)$, and verify that

$$[T]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}.$$

Theorem

Theorem: If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation with standard matrix A, and $\mathcal{B}$ is an ordered basis for $\mathbb{R}^n$, then the $\mathcal{B}$ matrix of T is

$$[T]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}.$$

Corollary: If A is diagonalizable with $D = P^{-1}AP$, and $\mathcal{B}$ is the basis of $\mathbb{R}^n$ consisting of the columns of P, then the $\mathcal{B}$ matrix of T is the diagonal matrix

$$[T]_{\mathcal{B}} = D.$$

Example

Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ such that the $\mathcal{B}$ matrix of T is diagonal where

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}.$$