November 16 MATH 1113 sec. 52 Fall 2018

Section 7.1: Fundamental Identities: Pythagorean, Sum, and Difference

We recall that the trigonometric functions are largely interrelated. For example, we already know that

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

An important property of this statement is that it is true for every real number x for which the sides of the equation are defined!

In this chapter, we will explore various relationships between the trigonometric functions.

Expressions and Types of Statements

To proceed, let's make a distinction between

- an expression,
- a statement,
- a conditional statement, and
- an identity.

Expression -vs- Statement

Definition: An expression is a grouping of numbers, symbols, and/or operators that may define a mathematical object or quantity. An expression is a *NOUN*.

Examples: Each of

$$x^2 + 7x, \quad \ln(\sin\theta), \quad 25 \div 7 \quad \text{and} \quad \frac{\sin(x)}{\cos(x)}$$

are expressions.

Expression -vs- Statement

Definition: A mathematical statement is an assertion that may be true or false. A statement is a *SENTENCE*.

Examples: Some examples of statements include

$$x^2 + 7x = 18, \qquad \sin(0^\circ) > 12, \qquad \cos\left(\frac{\pi}{2} - x\right) = \sin(x)$$

If $x > 0$, and $x < 0$, then x is a green eyed lion.

In a symbolic sentence, the verb is usually included in one of the symbols $=$, $\leq$, $\geq$, $>$, or $<$.

Conditional Statement

Definition: A conditional statement is a statement that is true under certain conditions. Typically, it is true when a variable is assigned a certain value/values. But it is false when other values are assigned.

For example:

$$x^2 + 7x = 18$$

is true if $x = 2$ or if $x = -9$. If any other value is assigned to x, the statement is false.

For example:

$$\cos(x) = \sin(x)$$

is true if $x = \frac{\pi}{4}$. It is also true if $x = \frac{5\pi}{4}$. In fact, it is true for infinitely many different values of x. However, it is **NOT ALWAYS** true. If $x = \frac{\pi}{2}$, then the statement is false.

Identity

Definition: An identity is a mathematical statement that is **ALWAYS** true. If an identity is stated as an equation, this means that for every value of any variable such that both sides of the equation are defined, both sides of the equation are the same.

For example:

$$\csc\left(\frac{\pi}{2} - x\right) = \sec(x)$$

is true for every real number x for which each side is defined.

For example:

$$\cos(-x) = \cos(x)$$

is true for every real number x for which each side is defined.

Identities We Already Know

Reciprocal: $\csc(x) = \dfrac{1}{\sin(x)}, \quad \sec(x) = \dfrac{1}{\cos(x)}, \quad , \cot(x) = \dfrac{1}{\tan(x)},$

$$\sin(x) = \frac{1}{\csc(x)}, \quad \cos(x) = \frac{1}{\sec(x)}, \quad , \tan(x) = \frac{1}{\cot(x)},$$

Quotient: $\tan(x) = \dfrac{\sin(x)}{\cos(x)}, \quad \cot(x) = \dfrac{\cos(x)}{\sin(x)}$

Identities We Already Know

Cofunction: $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$, $\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$,

$$\csc(x) = \sec\left(\frac{\pi}{2} - x\right), \quad \sec(x) = \csc\left(\frac{\pi}{2} - x\right),$$

$$\cot(x) = \tan\left(\frac{\pi}{2} - x\right), \quad \tan(x) = \cot\left(\frac{\pi}{2} - x\right).$$

Periodicity: $\sin(x + 2\pi) = \sin(x)$, $\cos(x + 2\pi) = \cos(x)$

$$\csc(x + 2\pi) = \csc(x), \quad \sec(x + 2\pi) = \sec(x)$$

$$\tan(x + \pi) = \tan(x), \quad \cot(x + \pi) = \cot(x)$$

Symmetry: $\sin(-x) = -\sin(x)$, $\cos(-x) = \cos(x)$, $\tan(-x) = -\tan(x)$

$$\csc(-x) = -\csc(x), \quad \sec(-x) = \sec(x), \quad \cot(-x) = -\cot(x).$$

Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

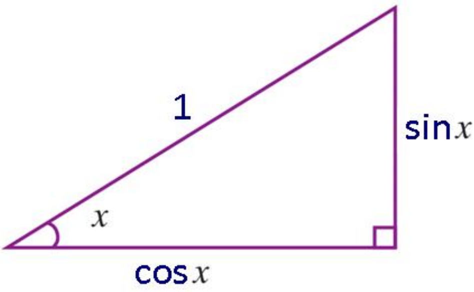


Figure: Triangle in a **unit** circle. This Pythagorean ID. follows directly from the Pythagorean Theorem.

Notation

To indicate a power n **different from -1** of a trigonometric function, it is standard to write

$$\sin^n(x), \quad \text{or} \quad \cos^n(x).$$

For example $\tan^3(x)$ is read

"the tangent cubed of x."

Note that

$$\tan^3(x) = (\tan(x))^3$$

We usually write this way for integer powers n. We NEVER write this for the power -1.

Pythagorean Identities

Use $\sin^2(x) + \cos^2(x) = 1$ along with other identities to show that

$$\tan^2(x) + 1 = \sec^2(x)$$

$\sin^2 x + \cos^2 x = 1$ Divide by $\cos^2(x)$ for $\cos(x) \neq 0$

$$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{\sin^2 x}{\cos^2 x} = \tan^2 x$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{1}{\cos^2 x} = \sec^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

Pythagorean Identities

We have the three new identities:

$$\sin^2(x) + \cos^2(x) = 1$$

$$\tan^2(x) + 1 = \sec^2(x), \quad \text{and}$$

$$1 + \cot^2(x) = \csc^2(x)$$

Note the the squares appearing here are NOT OPTIONAL! They are critical to the identities.

Question

The statement $\sin^2 x + \cos^2 x = 1$ is an identity. Which of the following statements are equivalent to this?

(a) $\sin^2 x = 1 - \cos^2 x$

(b) $\cos x = 1 - \sin x$

$\sqrt{a^2+b^2} \neq a+b$

(c) $|\cos x| = \sqrt{1 - \sin^2 x}$

(d) (a) and (b)

(e) (a) and (c)

Simplifying Expressions

Use the identities to simplify the expression (there may be more than one correct answer)

(a) $\dfrac{1}{\tan^2(x) + 1}$

$\tan^2 x + 1 = \sec^2 x$

$= \dfrac{1}{\sec^2 x}$

$= \cos^2 x$

(b) $\dfrac{\cot\alpha}{\csc\alpha}$

$\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$ $\quad$ $\csc\alpha = \dfrac{1}{\sin\alpha}$

$$= \dfrac{\dfrac{\cos\alpha}{\sin\alpha}}{\dfrac{1}{\sin\alpha}}$$

$$= \dfrac{\cos\alpha}{\sin\alpha} \cdot \dfrac{\sin\alpha}{1} = \cos\alpha$$

Question

The expression

$$\sin\theta(\csc\theta - \sin\theta)$$

is equivalent to which of the following?

(a) $\cos\theta$

(b) $\cos^2\theta$

(c) $1 - \csc\theta$

(d) $\sin^2\theta$

(e) none of the above is equivalent to the given expression

$= \sin\theta\csc\theta - \sin\theta\sin\theta$

$= 1 - \sin^2\theta$

$= \cos^2\theta$

Trigonometric Substitution

A very useful tool for Calculus is the art of **trigonometric substitution.** It involves writing algebraic expressions in terms of trigonometric ones!

Example: Assume $0 \leq \theta \leq \frac{\pi}{2}$. Use the substitution $\frac{x}{2} = \sin\theta$ to write $\sqrt{4 - x^2}$ as a trigonometric expression in θ.

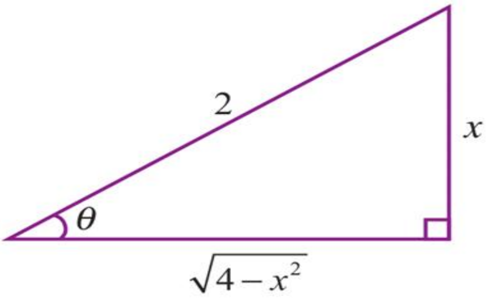


Figure: A representative triangle connecting x to θ.

From the triangle

$$\cos\theta = \frac{\sqrt{4-x^2}}{2}$$

So $$\sqrt{4-x^2} = 2\cos\theta$$

Sum and Difference Identities

Given two *angles* u and v, we wish to find a formula for $\cos(u - v)$.

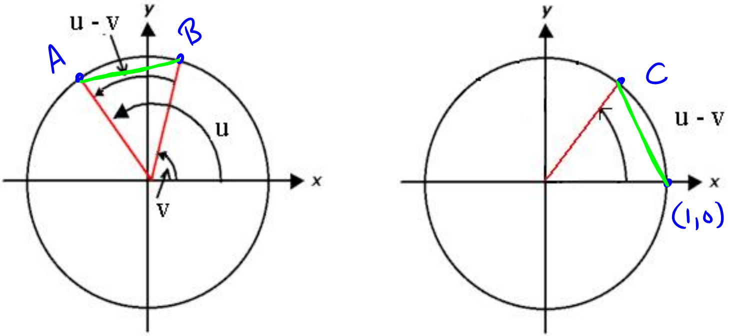


Figure: We construct the angle $u - v$ in a unit circle in two ways and equate the resulting chord lengths.

$A = (\cos u, \sin u)$ $B = (\cos v, \sin v)$ $C = (\cos(u-v), \sin(u-v))$

Derive the formula for $\cos(u - v)$

The line segments $\overline{AB}$ and $\overline{C(1,0)}$ have the same length.

$\overline{AB}$ squared

$$(\cos u - \cos v)^2 + (\sin u - \sin v)^2 =$$

$$\cos^2 u - 2\cos u \cos v + \cos^2 v + \sin^2 u - 2\sin u \sin v + \sin^2 v =$$

$$\cos^2 x + \sin^2 x = 1$$

$$2 - 2(\cos u \cos v + \sin u \sin v)$$

The distance between C and $(1,0)$ squared

$$(\cos(u-v)-1)^2+(\sin(u-v)-0)^2=$$

$$\cos^2(u-v)-2\cos(u-v)+1+\sin^2(u-v)=$$

$$2-2\cos(u-v)$$

Equating these lengths (squared)

$$2-2\cos(u-v)=2-2(\cos u\cos v+\sin u\sin v)$$

Subtract 2 then divide by -2

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

Difference of angles formula for

the Cosine.

Question

We know that $\cos(u-v) = \cos u \cos v + \sin u \sin v$. Use the fact that

$$\cos(u+v) = \cos(u-(-v)) = \cos u \cos(-v) + \sin u \sin(-v)$$

and the even/odd symmetry of the sine and cosine to deduce the sum of angles formula for the cosine. The result is

$$\cos(u+v) =$$

(a) $\cos u \sin v + \sin u \cos v$

$\cos(-v) = \cos v$

(b) $\cos u + \cos v$

$\sin(-v) = -\sin v$

(c) $\cos u \cos v - \sin u \sin v$

(d) $\cos u + \sin v$

Sum and Difference Identities

Cosine Identities:

(sum) $\cos(u+v) = \cos u \cos v - \sin u \sin v,$

(diff) $\cos(u-v) = \cos u \cos v + \sin u \sin v$

Sine Identities:

(sum) $\sin(u+v) = \sin u \cos v + \sin v \cos u$

(diff) $\sin(u-v) = \sin u \cos v - \sin v \cos u$

Tangent Identities:

(sum) $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v},$

(diff) $\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

Determine the Exact Value of Each Expression

$$\tan\left(\frac{5\pi}{12}\right)$$

$$\frac{5\pi}{12} = \frac{3\pi}{12} + \frac{2\pi}{12}$$

$$= \frac{\pi}{4} + \frac{\pi}{6}$$

Clear fractions

$$= \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1\cdot\frac{1}{\sqrt{3}}} \left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$

$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

Question

The exact value of $\sin\left(\frac{7\pi}{12}\right)$ is $\left(\text{hint: } \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}\right)$

(a) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

(b) $\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}$

(c) $\frac{\sqrt{3}-1}{2\sqrt{2}}$

(d) $\frac{1-\sqrt{3}}{2\sqrt{2}}$

Evaluate each expression using the given information.

Given: $\tan\alpha = -2, \quad \frac{\pi}{2} < \alpha < \pi$ quad II

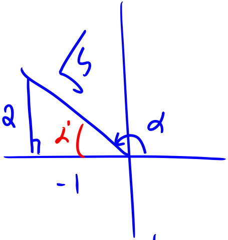
and $\sec\beta = \frac{4}{3}, \quad \frac{3\pi}{2} < \beta < 2\pi$ IV

$4^2 - 3^2 = 7$

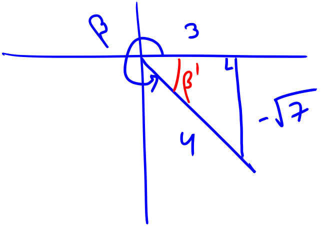
Draw diagrams

(a) $\tan(\alpha+\beta)$

(a) $\csc(\alpha+\beta)$



$\tan\alpha = -2$



$\tan\beta = \frac{-\sqrt{7}}{3}$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta} = \frac{-2 - \frac{\sqrt{7}}{3}}{1-(-2)\left(-\frac{\sqrt{7}}{3}\right)} \cdot \frac{3}{3}$$ clear fractions

$$= \frac{-6-\sqrt{7}}{3-2\sqrt{7}}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$= \frac{2}{\sqrt{5}} \cdot \frac{3}{4} + \left(\frac{-\sqrt{7}}{4}\right)\left(\frac{-1}{\sqrt{5}}\right) = \frac{6}{4\sqrt{5}} + \frac{\sqrt{7}}{4\sqrt{5}} = \frac{6+\sqrt{7}}{4\sqrt{5}}$$

$$\csc(\alpha+\beta) = \frac{4\sqrt{5}}{6+\sqrt{7}}$$