Nov. 16 Math 1190 sec. 51 Fall 2016

Section 5.3: The Fundamental Theorem of Calculus

Theorem: The Fundamental Theorem of Calculus (part 1)
If f is continuous on $[a, b]$ and the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{for} \quad a \le x \le b,$$

then g is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$g'(x) = f(x).$$

This means that the new function g is an **antiderivative** of f on (a, b)!
"FTC" = "fundamental theorem of calculus"

Question

Evaluate $\frac{d}{dx}\int_7^x \tan^{-1}(t)\,dt$

$f(t) = \tan^{-1} t$

so

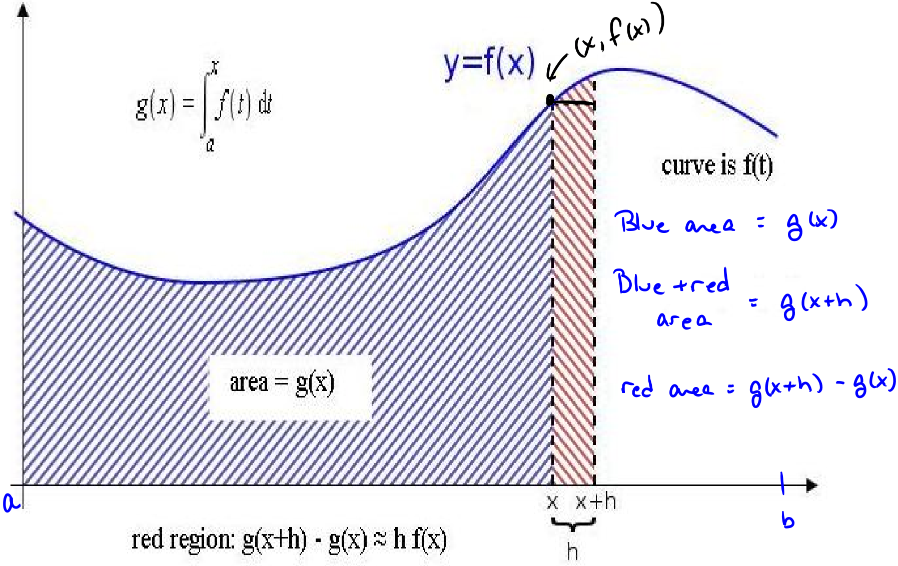
$f(x) = \tan^{-1} x$

(a) $\frac{1}{1+x^2}$

(b) $\tan^{-1}(x)$

(c) $\tan^{-1}(x) - \tan^{-1}(7)$

Geometric Argument of FTC



Recall $\quad g'(x) = \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h}$

height

width

red area $\approx$ red retangle area $= f(x) \cdot h$

So

$$g(x+h) - g(x) \approx f(x)h$$

This approximation becomes "more exact" as h gets smaller.

$$\frac{g(x+h) - g(x)}{h} \approx f(x)$$

Take $h \to 0$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

Chain Rule with FTC

Evaluate each derivative.

(a) $\dfrac{d}{dx}\displaystyle\int_0^{x^2} t^3\,dt$

Chain rule: If $u = u(x)$

$$\frac{d}{dx}F(u) = F'(u)\cdot\frac{du}{dx}$$

If $u = x^2$ and

$F(u) = \displaystyle\int_0^u t^3\,dt$: $\displaystyle\int_0^{x^2} t^3\,dt$ is our integral.

FTC $\rightarrow$ $F'(u) = u^3$ and $\dfrac{du}{dx} = 2x$ $\leftarrow$ power rule

$$\frac{d}{dx}\int_0^{x^2} t^3\,dt = F'(u)\frac{du}{dx} = u^3(2x) = (x^2)^3(2x) = 2x^7$$

(b) $\dfrac{d}{dx}\displaystyle\int_x^7 \cos(t^2)\,dt$

Recall

$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$

$$= \frac{d}{dx}\left(-\int_7^x \cos(t^2)\,dt\right)$$

$$= -\frac{d}{dx}\int_7^x \cos(t^2)\,dt$$

$$= -\cos(x^2)$$

Question

(c) $\frac{d}{dx}\int_3^{\sin x} \frac{1}{1+t^3}\,dt$

(a) $\frac{1}{1+\sin^3 x}$

(b) $\frac{\cos x}{1+\sin^3 x}$

(c) $\frac{\cos x}{1+x^3}$

(d) $\frac{-3\sin^2 x}{(1+\sin^3 x)^2}$

If $u = \sin x$ then $u' = \cos x$

$F(u) = \int_3^u \frac{1}{1+t^3}\,dt \Rightarrow F'(u) = \frac{1}{1+u^3}$

$\frac{d}{dx}\int_3^{\sin x} \frac{1}{1+t^3}\,dt = \frac{1}{1+(\sin x)^3} \cdot \cos x$

$= \frac{\cos x}{1+\sin^3 x}$

Theorem: The Fundamental Theorem of Calculus (part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$. (i.e. $F'(x) = f(x)$)

To evaluate $\int_a^b f(x)dx$, find an $F(x)$, then take the difference $F(b) - F(a)$.

Note: $\int_a^b f(x)dx$ is a number.

Example: Use the FTC to show that $\int_0^b x\,dx = \frac{b^2}{2}$

Here $f(x) = x = x^1$ ← one power

an antiderivative is $F(x) = \frac{x^{1+1}}{1+1} = \frac{x^2}{2}$

Note $F(b) = \frac{b^2}{2}$ and $F(0) = \frac{0^2}{2} = 0$

So

$$\int_0^b x\,dx = F(b) - F(0) = \frac{b^2}{2} - 0 = \frac{b^2}{2}$$

as expected !

Notation

Suppose F is an antiderivative of f. We write

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

or sometimes

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

For example

$$\int_0^b x \;\; dx = \left.\frac{x^2}{2}\right|_0^b = \frac{b^2}{2} - \frac{0^2}{2} = \frac{b^2}{2}$$

Evaluate each definite integral using the FTC

(a) $\int_0^2 3x^2\,dx = x^3 \Big|_0^2 = 2^3 - 0^3 = 8$

(b) $\int_{\frac{\pi}{2}}^{\pi} \cos x \, dx = \sin x \Big|_{\pi/2}^{\pi}$

$= \sin \pi - \sin \frac{\pi}{2}$

$= 0 - 1$

$= -1$

Question

(c) $\int_1^9 \frac{1}{2}u^{-1/2}\,du$ $= u^{1/2}\Big|_1^9 = 9^{1/2} - 1^{1/2} = 3 - 1 = 2$

(a) 8

(b) $\frac{13}{54}$

(c) 2

(d) $-\frac{1}{3}$

Note $\frac{1}{2}u^{-1/2} \xrightarrow[\text{antideriv.}]{\text{take}} \frac{1}{2}\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$

$\frac{1}{2}\frac{u^{1/2}}{1/2} = u^{1/2}$

(d) $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-t^2}}\,dt$ $= \sin^{-1} t \Big|_0^{1/2}$

$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$

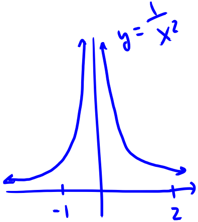
$= \frac{\pi}{6} - 0$

$= \pi/6$

Caveat! The FTC doesn't apply if f is not continuous!

The function $f(x) = \frac{1}{x^2}$ is positive everywhere on its domain. Now consider the calculation

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$$



Is this believable? Why or why not?

No, this "should" be true positive area,

but it's negative.

Question

Determine which, if any, of the following integrals **does not meet the criteria for the FTC to apply**.

(a) $\int_1^7 \ln(x)\,dx$

(b) $\int_1^e \ln(x)\,dx$

(c) $\int_0^7 \ln(x)\,dx$

$\ln x$ is not continuous on $[0, 7]$.

$\lim_{x \to 0^+} \ln x = -\infty$

(d) The FTC applies to all three of these.

An Observation

The $f(x)$ in the FTC is here replaced w/ $f'(x)$. So $F(x)$ is replaced with $f(x)$

If f is differentiable on $[a, b]$, note that

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This says that:

The integral of the **rate of change** of f over the interval $[a, b]$ is the **net change** of the function, $f(b) - f(a)$, over this interval.

Rectilinear Motion

If the position of a particle, relative to an origin, moving along a straight line is $s(t)$, then it's velocity is

$$v(t) = s'(t).$$

The net change result tells us that the net distance traveled on the time interval $[a, b]$, final position minus starting position, is

$$s(b) - s(a) = \int_a^b v(t)\,dt$$

We can say that the final position

$$s(b) = s(a) + \int_a^b v(t)\,dt.$$