Nov. 16 Math 1190 sec. 52 Fall 2016

Section 5.3: The Fundamental Theorem of Calculus

Theorem: The Fundamental Theorem of Calculus (part 1)
If f is continuous on $[a, b]$ and the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{for} \quad a \le x \le b,$$

then g is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$g'(x) = f(x).$$

This means that the new function g is an **antiderivative** of f on (a, b)!
"FTC" = "fundamental theorem of calculus"

Question

Evaluate $\dfrac{d}{dx}\displaystyle\int_7^x \tan^{-1}(t)\,dt$

(a) $\dfrac{1}{1+x^2}$

here $f(t) = \tan^{-1} t$

so

$f(x) = \tan^{-1} x$

(b) $\tan^{-1}(x)$

(c) $\tan^{-1}(x) - \tan^{-1}(7)$

Chain Rule with FTC

If f is continuous, $u(x)$ is some differentiable function of x and a is constant, then by the chain rule

$$\frac{d}{dx}\int_a^u f(t)\,dt = f(u)\frac{du}{dx}.$$

derivative of outside by the FTC

derivative of the inside

Example

Evaluate $\dfrac{d}{dx}\displaystyle\int_{\ln x}^{4} \frac{\sin^{-1} t}{t^2+1}\,dt$

$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$

also, if

$u = \ln x$ then $\dfrac{du}{dx} = \dfrac{1}{x}$

$$= \frac{d}{dx}\left(-\int_4^{\ln x} \frac{\sin^{-1} t}{t^2+1}\,dt\right)$$

$$= -\frac{d}{dx}\int_4^{\ln x} \frac{\sin^{-1} t}{t^2+1}\,dt$$

$$= -\frac{\sin^{-1}(\ln x)}{(\ln x)^2+1}\cdot\frac{1}{x} = \frac{-\sin^{-1}(\ln x)}{x\left((\ln x)^2+1\right)}$$

Question

Evaluate $\dfrac{d}{dx}\displaystyle\int_1^{e^{2x}} \ln(t^2+1)\,dt$

If $u = e^{2x}$, $\dfrac{du}{dx} = 2e^{2x}$

(a) $\frac{1}{e^{4x}+1}$

also $\left(e^{2x}\right)^2 = e^{4x}$

(b) $\frac{2e^{2x}}{e^{4x}+1}$

(c) $e^{2x}\ln(e^{4x}+1)$

(d) $2e^{2x}\ln(e^{4x}+1)$ $= 2e^{2x}\ln\left(\left(e^{2x}\right)^2+1\right)$

Theorem: The Fundamental Theorem of Calculus (part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$. (i.e. $F'(x) = f(x)$)

To evaluate $\int_a^b f(x)dx$ first find an antiderivative $F(x)$, then plug in b and a and compute $F(b) - F(a)$.

Note again $\int_a^b f(x)dx$ is a number.

Example: Use the FTC to show that $\int_0^b x\,dx = \frac{b^2}{2}$

Here $f(x) = x$ (x to the 1 power)

We can use antiderivative $F(x) = \frac{x^{1+1}}{1+1} = \frac{x^2}{2}$

$F(b) = \frac{b^2}{2}$ and $F(0) = \frac{0^2}{2} = 0$

So $\int_0^b x\,dx = F(b) - F(0) = \frac{b^2}{2} - 0 = \frac{b^2}{2}$

Notation

Suppose F is an antiderivative of f. We write

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

or sometimes

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

For example

$$\int_0^b x\;\;dx = \left.\frac{x^2}{2}\right|_0^b = \frac{b^2}{2} - \frac{0^2}{2} = \frac{b^2}{2}$$

Evaluate each definite integral using the FTC

(a) $\int_0^2 3x^2\,dx = x^3 \Big|_0^2 = 2^3 - 0^3 = 8$

Note $3x^2 \xrightarrow{\text{take antiderivative}} 3 \cdot \frac{x^{2+1}}{2+1} = 3\,\frac{x^3}{3} = x^3$

(b) $\int_{\frac{\pi}{2}}^{\pi} \cos x \, dx$ $= \sin x \Big|_{\pi/2}^{\pi}$

$= \sin \pi - \sin \frac{\pi}{2}$

$= 0 - 1$

$= -1$

Question

(c) $\int_1^9 \frac{1}{2}u^{-1/2}\,du$ $= u^{1/2}\Big|_1^9 = 9^{1/2} - 1^{1/2} = 3-1 = 2$

$\frac{1}{2}u^{-1/2} \xrightarrow{\text{take antider.}} \frac{1}{2}\,\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{1}{2}\,\frac{u^{1/2}}{1/2} = u^{1/2}$

(a) 8

(b) $\frac{13}{54}$

(c) 2

(d) $-\frac{1}{3}$

(d) $\int_0^{1/2} \frac{1}{\sqrt{1-t^2}}\, dt = \sin^{-1} t \Big|_0^{1/2}$

$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$

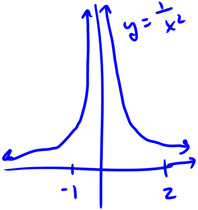
$= \frac{\pi}{6} - 0$

$= \frac{\pi}{6}$

Caveat! The FTC doesn't apply if f is not continuous!

The function $f(x) = \frac{1}{x^2}$ is positive everywhere on its domain. Now consider the calculation

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$$



Is this believable? Why or why not?

The answer isn't believable. The integral of a positive function should be positive.

Question

Determine which, if any, of the following integrals **does not meet the criteria for the FTC to apply**.

(a) $\int_1^7 \ln(x)\,dx$

(b) $\int_1^e \ln(x)\,dx$

(c) $\int_0^7 \ln(x)\,dx$

ln x is not continuous on [0, 7], it's not defined @ 0.

$$\lim_{x \to 0^+} \ln x = -\infty$$

(d) The FTC applies to all three of these.

An Observation

If f is differentiable on $[a, b]$, note that

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This says that:

The integral of the **rate of change** of f over the interval $[a, b]$ is the **net change** of the function, $f(b) - f(a)$, over this interval.

Rectilinear Motion

If the position of a particle, relative to an origin, moving along a straight line is $s(t)$, then it's velocity is

$$v(t) = s'(t).$$

The net change result tells us that the net distance traveled on the time interval $[a, b]$, final position minus starting position, is

$$s(b) - s(a) = \int_a^b v(t)\,dt$$

We can say that the final position

$$s(b) = s(a) + \int_a^b v(t)\,dt.$$

Example

A ball is dropped from a 300 ft cliff. It's velocity is $v(t) = -32t$ ft/sec. Determine the height of the ball after 2 seconds.

Calling the height $s(t)$ (in feet), then

$$s(2) - s(0) = \int_0^2 v(t)\,dt \Rightarrow s(2) = s(0) + \int_0^2 v(t)\,dt$$

$s(0) = 300$ ft

So

$$s(2) = 300 + \int_0^2 (-32t)\,dt = 300 + \left(-32 \cdot \frac{t^2}{2}\right|_0^2$$

$$= 300 + \left[-16 \cdot 2^2 - (-16 \cdot 0^2)\right] = 300 - 64 = 236$$

It's height is 236 ft.

Section 5.4: Properties of the Definite Integral

Suppose that f and g are integable on $[a, b]$ and let k be constant.

I. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

II. $\displaystyle\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

II. $\displaystyle\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Examples

Suppose $\int_1^4 f(x)\,dx = 3$ and $\int_1^4 g(x)\,dx = -7$. Evaluate

(i) $\displaystyle\int_1^4 -2f(x)\,dx$ $= -2\displaystyle\int_1^4 f(x)\,dx = -2(3) = -6$

property I.

(ii) $\displaystyle\int_1^4 [f(x)+3g(x)]\,dx$ $= \displaystyle\int_1^4 f(x)\,dx + \int_1^4 3g(x)\,dx$ ← property II

$= \displaystyle\int_1^4 f(x)\,dx + 3\int_1^4 g(x)\,dx$ ← property I

$= 3 + 3\cdot(-7) = 3-21 = -18$

Question

Suppose $\int_1^4 f(x)\,dx = 3$ and $\int_1^4 g(x)\,dx = -7$. Evaluate

$$\int_1^4 [g(x) - 3f(x)]\,dx$$

$$= \int_1^4 g(x)\,dx \quad - 3\int_1^4 f(x)\,dx$$

$$= -7 \quad - 3 \cdot 3 \quad = -7 - 9 \quad = -16$$

(a) 16

(b) -16

(c) -2

(d) 2