November 16 Math 2306 sec 51 Fall 2015

Section 11.1: (Brief Overview of Inner Product and Orthogonality)

Suppose two functions f and g are integrable on the interval $[a, b]$. We define the **inner product** of f and g on $[a, b]$ as

$$< f, g >= \int_a^b f(x)g(x)\,dx.$$

We say that f and g are **orthogonal** on $[a, b]$ if

$$< f, g >= 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

Orthogonal Set

A set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$< \phi_m, \phi_n >= \int_a^b \phi_m(x)\phi_n(x)\,dx = 0 \quad \text{whenever} \quad m \neq n.$$

Note that any function $\phi(x)$ that is not identically zero will satisfy

$$< \phi, \phi >= \int_a^b \phi^2(x)\,dx > 0.$$

Hence we define the **square norm** of ϕ (on $[a, b]$) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x)\,dx}.$$

An Orthogonal Set of Functions

The set $\{1, \cos(nx), \sin(mx) | \text{for integers } n, m \geq 1\}$ is orthogonal on $[-\pi, \pi]$. Moreover, we have the properties

$$\int_{-\pi}^{\pi} \cos nx \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx \, dx = 0 \quad \text{for all} \quad n, m \geq 1,$$

$$\int_{-\pi}^{\pi} \cos nx \, \sin mx \, dx = 0 \quad \text{for all} \quad m, n \geq 1,$$

$$\langle 1, 1 \rangle = \int_{-\pi}^{\pi} dx = x \Big|_{-\pi}^{\pi}$$

$$= \pi - (-\pi) = 2\pi$$

$$\int_{-\pi}^{\pi} \cos nx \, \cos mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases},$$

$$\int_{-\pi}^{\pi} \sin nx \, \sin mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases}.$$

Section 11.2: Fourier Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines.**

Task: Find coefficients (numbers) $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that[1]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

[1] We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

For a known function f defined on $(-\pi, \pi)$, assume the series holds. Find the coefficient b_4. Multiply both sides by $\sin 4x$

$$f(x)\sin 4x = \frac{a_0}{2}\sin 4x + \sum_{n=1}^{\infty}\left(a_n\ \cos nx \sin 4x + b_n\ \sin nx \sin 4x\right).$$

Now integrate both sides with respect to x from $-\pi$ to π (assume it is valid to integrate first and sum later).

$$\int_{-\pi}^{\pi} f(x)\sin 4x\,dx = \int_{-\pi}^{\pi}\frac{a_0}{2}\sin 4x\,dx \quad +$$

$$\sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n\ \cos nx \sin 4x\,dx + \int_{-\pi}^{\pi} b_n\ \sin nx \sin 4x\,dx\right).$$

$$\int_{-\pi}^{\pi} f(x)\sin 4x\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \overset{=0}{\sin 4x\,dx} \quad +$$

$$\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi} \overset{=0}{\cos nx\sin 4x\,dx} + b_n\int_{-\pi}^{\pi} \sin nx\sin 4x\,dx\right).$$

Note $\int_{-\pi}^{\pi} \sin(4x)\,dx = 0$

also $\int_{-\pi}^{\pi} \cos(nx)\sin(4x)\,dx = 0$ for all n

So far we have

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\sin(4x)\,dx$$

But $\displaystyle\int_{-\pi}^{\pi} \sin(nx)\sin(4x)\,dx = \begin{cases} 0, & n \neq 4 \\ \pi, & n = 4 \end{cases}$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = b_4\,\pi$$

$$\Rightarrow \quad b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx$$

The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

Where

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx
\end{aligned}$$

Example

Find the Fourier series of the piecewise defined function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right|_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right]$$

By parts

$u = x \qquad du = dx$

$dv = \cos(nx)\,dx$

$v = \frac{1}{n}\sin(nx)$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\underbrace{\sin(n\pi)}_{=0} - 0 + \frac{1}{n^2}\cos(nx)\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\,\frac{1}{n^2}\left(\cos(n\pi) - \cos(0)\right)$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

Note

$\cos(n\pi) = (-1)^n$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right]$$

By parts

$u = x \qquad du = dx$

$dv = \sin(nx)\,dx$

$v = \frac{-1}{n}\cos(nx)$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}\cos(n\pi) - 0\right] + \frac{1}{n^2}\sin(nx)\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{-\pi}{n}(-1)^n\right) + \frac{1}{\pi n^2}\left(\underbrace{\sin(n\pi)}_{=0} - \sin(0)\right)$$

$$= \frac{-(-1)^n}{n} = \frac{(-1)^{n+1}}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos(nx) + \frac{(-1)^{n+1}}{n} \sin(nx) \right]$$

Fourier Series on an interval $(-p, p)$

The set of functions The set $\{1, \cos\left(\frac{n\pi x}{p}\right), \sin\left(\frac{m\pi x}{p}\right) | n, m \geq 1\}$ is orthogonal on $[-p, p]$. Moreover, we have the properties

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\, dx = 0 \quad \text{and} \quad \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \;\; \text{for all} \;\; n, m \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \quad \text{for all} \quad m, n \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \cos\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases},$$

$$\int_{-p}^{p} \sin\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases}.$$

Fourier Series on an interval $(-p, p)$

The orthogonality relations provide for an expansion of a function f defined on $(-p, p)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right)$$

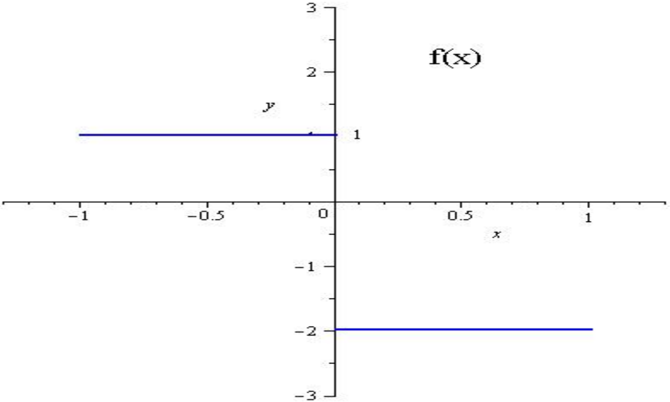
where

$$\begin{aligned} a_0 &= \frac{1}{p} \int_{-p}^{p} f(x)\, dx, \\ a_n &= \frac{1}{p} \int_{-p}^{p} f(x) \cos\left(\frac{n\pi x}{p}\right) dx, \quad \text{and} \\ b_n &= \frac{1}{\pi} \int_{-p}^{p} f(x) \sin\left(\frac{n\pi x}{p}\right) dx \end{aligned}$$

Find the Fourier series of f

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}$$

Here , $p = 1$



$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} dx + \int_{0}^{1}(-2)\,dx$$

$$= x\Big|_{-1}^{0} - 2x\Big|_{0}^{1} = (0-(-1)) - 2(1-0) = -1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)dx = \int_{-1}^{0}\cos(n\pi x)\,dx - \int_{0}^{1} 2\cos(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} - \frac{2}{n\pi}\sin(n\pi x)\Big|_{0}^{1}$$

$$= \frac{1}{n\pi}\left[\sin 0 - \sin(-n\pi)\right] - \frac{2}{n\pi}\left[\sin(n\pi) - \sin 0\right] = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\,\sin\left(\frac{n\pi x}{1}\right)dx = \int_{-1}^{0} \sin(n\pi x)\,dx - \int_{0}^{1} 2\sin(n\pi x)\,dx$$

$$= \frac{-1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0} + \frac{2}{n\pi}\cos(n\pi x)\Big|_{0}^{1}$$

$$= \frac{-1}{n\pi}\left[\cos 0 - \cos(-n\pi)\right] + \frac{2}{n\pi}\left[\cos(n\pi) - \cos 0\right]$$

$$= \frac{-1}{n\pi} + \frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n\pi} - \frac{2}{n\pi} = \frac{3}{n\pi}\left((-1)^n - 1\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

$$f(x) = \frac{-1}{2} + \sum_{n=1}^{\infty} \frac{3}{n\pi}\left((-1)^n - 1\right) \sin(n\pi x)$$