November 16 Math 2306 sec. 53 Fall 2018

Section 17: Fourier Series: Trigonometric Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines**.

Task: Find coefficients (numbers) $a_0, a_1, a_2, \dots$ and $b_1, b_2, \dots$ such that[1]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

[1] We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

Orthogonal Set of Functions

The set $\{1, \cos(nx), \sin(nx) \,|\, n = 1, 2, 3, \ldots\}$ is orthogonal on the interval $[-\pi, \pi]$.

This means that if we consider two functions $\phi_1(x)$ and $\phi_2(x)$ in this set, then

$$\int_{-\pi}^{\pi} \phi_1(x)\phi_2(x)\, dx = \begin{cases} 0, & \phi_1 \neq \phi_2 \\ \pi, & \phi_1 = \phi_2 \end{cases} \quad \text{and } \phi_1(x) \neq 1$$

$$\int_{-\pi}^{\pi} 1 \cdot 1 \, dx = 2\pi$$

Finding an Example Coefficient

For a known function f defined on $(-\pi, \pi)$, assume there is such a series[2]. Let's find the coefficient b_4.

We start by assuming that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

Then we multiplied through by $\sin 4x$

$$f(x) {\color{red}\sin 4x} = \frac{a_0}{2} {\color{red}\sin 4x} + \sum_{n=1}^{\infty} \left(a_n \cos nx {\color{red}\sin 4x} + b_n \sin nx {\color{red}\sin 4x}\right).$$

[2] We will also assume that the order of integrating and summing can be interchanged.

Finding an Example Coefficient

Then we integrate each side from $-\pi$ to π assuming that it is OK to interchange the order of summation and integration.

$$\int_{-\pi}^{\pi} f(x)\sin 4x\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \sin 4x\,dx \quad +$$

$$\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi} \cos nx \sin 4x\,dx + b_n\int_{-\pi}^{\pi} \sin nx \sin 4x\,dx\right).$$

Now we use the known orthogonality property. Recall that $\int_{-\pi}^{\pi} \sin 4x\,dx = 0$, and that for every $n = 1, 2, \ldots$

$$\int_{-\pi}^{\pi} \cos nx \sin 4x\,dx = 0$$

So the constant and all cosine terms are gone leaving

$$\int_{-\pi}^{\pi} f(x)\sin 4x\,dx = \sum_{n=1}^{\infty}\left(b_n\int_{-\pi}^{\pi}\sin nx\sin 4x\,dx\right).$$

But we also know that

$$\int_{-\pi}^{\pi}\sin nx\sin 4x\,dx = 0, \quad \text{for } n\neq 4, \text{ and} \quad \int_{-\pi}^{\pi}\sin 4x\sin 4x\,dx = \pi.$$

Hence the sum reduces to the single term

$$\int_{-\pi}^{\pi} f(x)\sin 4x\,dx = \pi b_4$$

from which we determine

$$b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 4x\,dx.$$

Finding Fourier Coefficients

Note that there was nothing special about seeking the 4^{th} sine coefficient b_4. We could have just as easily sought b_m for any positive integer m. We would simply start by introducing the factor $\sin(mx)$.

Moreover, using the same orthogonality property, we could pick on the a's by starting with the factor $\cos(mx)$—including the constant term since $\cos(0 \cdot x) = 1$. The only minor difference we want to be aware of is that

$$\int_{-\pi}^{\pi} \cos^2(mx)\,dx = \begin{cases} 2\pi, & m = 0 \\ \pi, & m \geq 1 \end{cases}$$

Careful consideration of this sheds light on why it is conventional to take the constant to be $\frac{a_0}{2}$ as opposed to just a_0.

The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

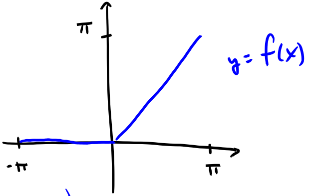
Where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}$$

Example

Find the Fourier series of the piecewise defined function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

Using the formulas

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx \right)$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x\,dx\right) = \frac{1}{\pi}\left(\frac{x^2}{2}\right)\Big|_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$\boxed{a_0 = \frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cdot\cos(nx)\,dx + \int_{0}^{\pi} x\cos(nx)\,dx\right)$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx) - \frac{1}{n^2}\cos(nx)\right]\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\sin(n\pi) - \frac{1}{n^2}\cos(n\pi) - \left(0 - \frac{1}{n^2}\cos(0)\right)\right]$$

$$\sin(n\pi) = 0 \quad \text{for} \quad n = 1, 2, 3, \dots$$

$$\cos(n\pi) = \begin{cases} 1, & n\text{-even} \\ -1, & n\text{-odd} \end{cases} = (-1)^n$$

$$a_n = \frac{1}{\pi}\left(\frac{-1}{n^2}(-1)^n + \frac{1}{n^2}\right) = \frac{1-(-1)^n}{n^2\pi}$$

$$\boxed{a_n = \frac{1-(-1)^n}{n^2\pi}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cdot\sin(nx)\,dx + \int_{0}^{\pi} x\sin(nx)\,dx\right)$$

$$= \frac{1}{\pi}\left(-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{-\pi}{n}\cos(n\pi) + \frac{1}{n^2}\underbrace{\sin(n\pi)}_{0} - \left(0 + \frac{1}{n^2}\underbrace{\sin(0)}_{0}\right)\right)$$

$$b_n = \frac{1}{\pi}\left(\frac{-\pi}{n}\right)\cos(n\pi) = \frac{-(-1)^n}{n} = \frac{(-1)^{n+1}}{n}$$

$$b_n = \frac{(-1)^{n+1}}{n}$$

$a_0 = \frac{\pi}{2}$ so $\frac{a_0}{2} = \frac{\pi}{4}$

$$\boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right)}$$

Fourier Series on an interval $(-p, p)$

The set of functions $\{1, \cos\left(\frac{n\pi x}{p}\right), \sin\left(\frac{m\pi x}{p}\right) | n, m \geq 1\}$ is orthogonal on $[-p, p]$. Moreover, we have the properties

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\, dx = 0 \quad \text{and} \quad \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \;\text{ for all }\; n, m \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \quad \text{for all} \quad m, n \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \cos\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases},$$

$$\int_{-p}^{p} 1 \cdot 1\, dx = 2p$$

$$\int_{-p}^{p} \sin\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases}.$$

Fourier Series on an interval $(-p, p)$

The orthogonality relations provide for an expansion of a function f defined on $(-p, p)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right)$$

where

$$\begin{aligned}
a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx, \\
a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx, \quad \text{and} \\
b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx
\end{aligned}$$