

Section 11.1: (Brief Overview of Inner Product and Orthogonality)

Suppose two functions f and g are integrable on the interval $[a, b]$. We define the **inner product** of f and g on $[a, b]$ as

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx.$$

We say that f and g are **orthogonal** on $[a, b]$ if

$$\langle f, g \rangle = 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

Orthogonal Set

A set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x) \phi_n(x) dx = 0 \quad \text{whenever} \quad m \neq n.$$

Note that any function $\phi(x)$ that is not identically zero will satisfy

$$\langle \phi, \phi \rangle = \int_a^b \phi^2(x) dx > 0.$$

Hence we define the **square norm** of ϕ (on $[a, b]$) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x) dx}.$$

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \dots, \sin x, \sin 2x, \sin 3x, \dots\} \quad \text{on} \quad [-\pi, \pi].$$

We've already shown that $\langle 1, \cos(nx) \rangle = 0$, $\langle 1, \sin(mx) \rangle = 0$ and $\langle \cos(nx), \sin(mx) \rangle = 0$ for each integer $n, m \geq 1$.

An Orthogonal Set of Functions continued...

Use the identity

$$\cos nx \cos mx = \frac{1}{2}(\cos((n+m)x) + \cos((n-m)x))$$

to show that

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = 0 \quad \text{whenever } n \neq m.$$

$$\begin{aligned}\langle \cos(nx), \cos(mx) \rangle &= \int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((n+m)x) + \cos((n-m)x)) dx \\ &= \frac{1}{2} \left[\frac{1}{n+m} \sin((n+m)x) + \frac{1}{n-m} \sin((n-m)x) \right] \Big|_{-\pi}^{\pi}\end{aligned}$$

$$= \frac{1}{2} \frac{1}{n+m} \left[\sin((n+m)\pi) - \sin(-(n+m)\pi) \right]$$

$$+ \frac{1}{2} \frac{1}{n-m} \left[\sin((n-m)\pi) - \sin(-(n-m)\pi) \right]$$

$$= 0$$

Evaluate

$$\int_{-\pi}^{\pi} \cos nx \cos nx dx = \int_{-\pi}^{\pi} \cos^2(nx) dx$$

$$= \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2} \cos(2nx) \right) dx$$

$$= \frac{1}{2} \times \left[\frac{\pi}{\pi} + \frac{1}{4n} \sin(2nx) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2} (\pi - (-\pi)) + \frac{1}{4n} \sin(2n\pi) - \frac{1}{4n} \sin(-2n\pi) = \pi$$

An Orthogonal Set of Functions

The set $\{1, \cos(nx), \sin(mx) | \text{for integers } n, m \geq 1\}$ is orthogonal on $[-\pi, \pi]$. Moreover, we have the properties

$$\int_{-\pi}^{\pi} \cos nx \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx \, dx = 0 \quad \text{for all } n, m \geq 1,$$

$$\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0 \quad \text{for all } m, n \geq 1,$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases},$$

$$\begin{aligned} \langle 1, 1 \rangle &= \int_{-\pi}^{\pi} 1 \, dx \\ &= x \Big|_{-\pi}^{\pi} = \pi - (-\pi) \end{aligned}$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases}.$$

Section 11.2: Fourier Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines**.

Task: Find coefficients (numbers) a_0, a_1, a_2, \dots and b_1, b_2, \dots such that¹

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

¹We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

For a known function f defined on $(-\pi, \pi)$, assume the series holds.
Find the coefficient b_4 . Multiply both sides by $\sin 4x$

$$f(x) \sin 4x = \frac{a_0}{2} \sin 4x + \sum_{n=1}^{\infty} (a_n \cos nx \sin 4x + b_n \sin nx \sin 4x).$$

Now integrate both sides with respect to x from $-\pi$ to π (assume it is valid to integrate first and sum later).

$$\int_{-\pi}^{\pi} f(x) \sin 4x \, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \sin 4x \, dx + \sum_{n=1}^{\infty} \left(\int_{-\pi}^{\pi} a_n \cos nx \sin 4x \, dx + \int_{-\pi}^{\pi} b_n \sin nx \sin 4x \, dx \right).$$

$$\int_{-\pi}^{\pi} f(x) \sin 4x \, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \sin 4x \, dx +$$

$$\sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \sin 4x \, dx + b_n \int_{-\pi}^{\pi} \sin nx \sin 4x \, dx \right).$$

From before $\int_{-\pi}^{\pi} \sin(4x) \, dx = 0$

$$\int_{-\pi}^{\pi} \cos(nx) \sin(4x) \, dx = 0$$

So far we have

$$\int_{-\pi}^{\pi} f(x) \sin(4x) dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx) \sin(4x) dx$$

But $\int_{-\pi}^{\pi} \sin(nx) \sin(4x) dx = \begin{cases} 0, & n \neq 4 \\ \pi, & n = 4 \end{cases}$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \sin(4x) dx = b_4 \pi \Rightarrow$$

$$b_4 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(4x) dx$$

The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad \text{and}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Example

Find the Fourier series of the piecewise defined function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \leq x < \pi \end{cases}$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 0 dx + \frac{1}{\pi} \int_0^{\pi} x dx \\ &= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{1}{\pi} \left(\frac{\pi^2}{2} - 0 \right) = \frac{\pi}{2} \end{aligned}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \int_{-\pi}^0 0 \cdot \cos(nx) dx + \frac{1}{\pi} \int_0^{\pi} x \cos(nx) dx$$

$$\begin{aligned}
 &= \frac{1}{\pi} \left[\frac{x}{n} \sin(nx) \right]_0^\pi - \frac{1}{n} \int_0^\pi \sin(nx) dx \quad \text{By parts} \\
 &= \frac{1}{\pi} \left[\frac{\pi}{n} \sin(n\pi) - 0 \right] + \frac{1}{\pi} \frac{1}{n^2} (\cos(nx)) \Big|_0^\pi \\
 &= \frac{1}{\pi n^2} (\cos(n\pi) - \cos 0)
 \end{aligned}$$

$u = x \quad du = dx$
 $dv = \cos(nx) dx$
 $v = \frac{1}{n} \sin(nx)$

Recall

$$= \frac{(-1)^n - 1}{\pi n^2}$$

$$\cos(n\pi) = (-1)^n$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_{-\pi}^0 0 \sin(nx) dx + \frac{1}{\pi} \int_0^{\pi} x \sin(nx) dx$$

$$= \frac{1}{\pi} \left[-\frac{x}{n} \cos(nx) \right]_0^\pi + \frac{1}{n} \int_0^\pi \cos(nx) dx$$

B_n part

$$u=x, \quad du=dx$$

$$= \frac{1}{\pi} \left[-\frac{\pi}{n} \cos(n\pi) - 0 \right] + \frac{1}{\pi} \frac{1}{n^2} \sin(nx) \Big|_0^\pi$$

$$du = \sin(nx) dx$$

$$v = -\frac{1}{n} \cos(nx)$$

$$= -\frac{1}{n} (-1)^n + \frac{1}{\pi n^2} (\sin(n\pi) - \sin 0)$$

$$= \frac{(-1)^{n+1}}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos(nx) + \frac{(-1)^{n+1}}{n} \sin(nx) \right]$$