Nov. 18 Math 1190 sec. 51 Fall 2016

Section 5.3: The Fundamental Theorem of Calculus

Theorem: The Fundamental Theorem of Calculus (part 2)
If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$. (i.e. $F'(x) = f(x)$)

To use the FTC, we
(i) find an antiderivative F for the integrand, then
(ii) evaluate the difference $F(b) - F(a)$.

Evaluate the definite integral using the FTC

$$\int_{-e^2}^{-2} \frac{ds}{s} = \int_{-e^2}^{-2} \frac{1}{s}\,ds = \ln|s| \Big|_{-e^2}^{-2}$$

$$= \ln|-2| - \ln|-e^2|$$

$$= \ln 2 - \ln e^2$$

$$= \ln 2 - 2\ln e = \ln 2 - 2$$

Question

Given that $\frac{d}{dx}xe^x = e^x + xe^x$, evaluate the integral

$$\int_1^2 (e^x + xe^x)\,dx \quad = xe^x \Big|_1^2 \quad = 2e^2 - 1e^1 \quad = 2e^2 - e$$

(a) $2e^2$

(b) $2e^2 - e$

(c) xe^x

(d) $3e^2 - 2e$

Finding Area

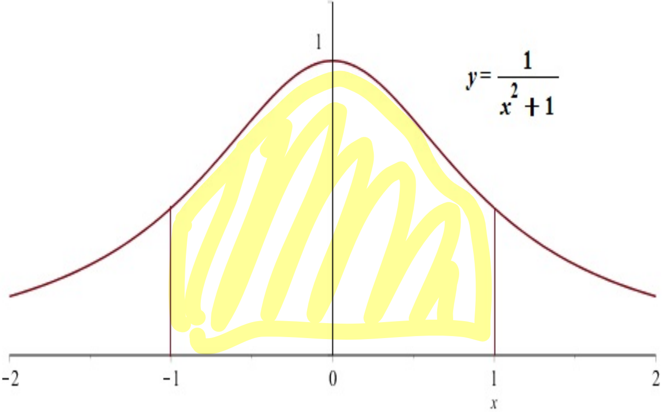

Figure: Use the FTC to find the area of the region under the graph of $y = \frac{1}{1+x^2}$, bounded below by the x-axis, and between the vertical lines $x = -1$ and $x = 1$.

Since $y = \frac{1}{x^2+1}$ is positive,

$$\text{Area} = \int_{-1}^{1} \frac{1}{x^2+1}\, dx$$

$$= \tan^{-1} x \Big|_{-1}^{1} = \tan^{-1} 1 - \tan^{-1}(-1)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

An Observation

If f is differentiable on $[a, b]$, note that

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This says that:

The integral of the **rate of change** of f over the interval $[a, b]$ is the **net change** of the function, $f(b) - f(a)$, over this interval.

Rectilinear Motion

If the position of a particle, relative to an origin, moving along a straight line is $s(t)$, then it's velocity is

$$v(t) = s'(t).$$

The net change result tells us that the net distance traveled on the time interval $[a, b]$, final position minus starting position, is

$$s(b) - s(a) = \int_a^b v(t)\,dt$$

We can say that the final position

$$s(b) = s(a) + \int_a^b v(t)\,dt.$$

Example

A ball is dropped from a 300 ft cliff. It's velocity is $v(t) = -32t$ ft/sec. Determine the height of the ball after 2 seconds.

Letting $s(t)$ be the height in ft. (relative to the ground)

$s(0) = 300$ ft

v in ft/sec

dt in sec

$$s(2) = s(0) + \int_0^2 v(t)\,dt$$

$$= 300 + \int_0^2 (-32t)\,dt = 300 + \left(-16t^2\right|_0^2$$

$$= 300 + \left(-16\cdot 2^2 - (-16\cdot 0^2)\right) = 300 - 64 = 236$$

The height is 236 ft @ 2 seconds.

Section 5.4: Properties of the Definite Integral

Suppose that f and g are integable on $[a, b]$ and let k be constant.

I. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

II. $\displaystyle\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

II. $\displaystyle\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Examples

Suppose $\int_1^4 f(x)\,dx = 3$ and $\int_1^4 g(x)\,dx = -7$. Evaluate

(i) $\displaystyle\int_1^4 -2f(x)\,dx$ $\displaystyle = -2\int_1^4 f(x)dx = -2(3) = -6$

property I

(ii) $\displaystyle\int_1^4 [f(x)+3g(x)]\,dx$ $\displaystyle = \int_1^4 f(x)\,dx + \int_1^4 3g(x)\,dx$ property II

$\displaystyle = \int_1^4 f(x)\,dx + 3\int_1^4 g(x)\,dx$ property I

$= 3 + 3(-7) = 3-21 = -18$

Question

Suppose $\int_1^4 f(x)\,dx = 3$ and $\int_1^4 g(x)\,dx = -7$. Evaluate

$$\int_1^4 [g(x) - 3f(x)]\,dx \quad = \int_1^4 g(x)\,dx - 3\int_1^4 f(x)\,dx$$

$$= -7 - 3 \cdot 3 = -16$$

(a) 16

(b) -16

(c) -2

(d) 2

The Sum/Difference in General

If $f_1, f_2, \ldots, f_n$ are integrable on $[a, b]$ and $k_1, k_2, \ldots, k_n$ are constants, then

$$\int_a^b [k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)]\, dx =$$

$$k_1 \int_a^b f_1(x)\, dx + k_2 \int_a^b f_2(x)\, dx + \cdots + k_n \int_a^b f_n(x)\, dx$$

Example

Evaluate $\displaystyle\int_1^2 \frac{x^3+2x^2+4}{x}\,dx$

$$= \int_1^2 \left(\frac{x^3}{x} + \frac{2x^2}{x} + \frac{4}{x}\right)dx = \int_1^2 \left(x^2 + 2x + \frac{4}{x}\right)dx$$

$$= \int_1^2 x^2\,dx + 2\int_1^2 x\,dx + 4\int_1^2 \frac{1}{x}\,dx$$

$$= \frac{x^3}{3}\Big|_1^2 + \cancel{2}\frac{x^2}{\cancel{2}}\Big|_1^2 + 4\ln|x|\,\Big|_1^2 = \left(\frac{2^3}{3} - \frac{1^3}{3}\right) + \left(2^2 - 1^2\right) + \left(4\ln 2 - 4\ln 1\right)$$

$$= \frac{8}{3} - \frac{1}{3} + 4 - 1 + 4\ln 2 - 0 = \frac{7}{3} + 3 + 4\ln 2$$

$$= \frac{16}{3} + 4\ln 2$$

Question

Evaluate $\int_0^1 (3x^2 + 4x - 2)\,dx$ $= x^3 + 2x^2 - 2x \Big|_0^1$

$= 1 + 2 - 2 - (0 + 0 - 0)$

$= 1$

(a) 5

(b) $x^3 + 2x^2 - 2x$

(c) 1

(d) $6x + 4$