# Nov. 18 Math 1190 sec. 52 Fall 2016

## Section 5.3: The Fundamental Theorem of Calculus

**Theorem: The Fundamental Theorem of Calculus (part 2)**
If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$ on $[a, b]$. (i.e. $F'(x) = f(x)$)

To use the FTC, we
(i) find an antiderivative $F$ for the integrand, then
(ii) evaluate the difference $F(b) - F(a)$.

# Evaluate the definite integral using the FTC

$$\int_{-e^2}^{-2} \frac{ds}{s} = \int_{-e^2}^{-2} \frac{1}{s}\, ds = \ln|s| \Big|_{-e^2}^{-2}$$

$$= \ln|-2| - \ln|-e^2|$$

$$= \ln 2 - \ln e^2$$

$$= \ln 2 - 2\ln e$$

$$= \ln 2 - 2 \cdot 1$$

$$= \ln 2 - 2$$

# Question

Given that $\frac{d}{dx}xe^x = e^x + xe^x$, evaluate the integral

$$\int_1^2 (e^x + xe^x)\, dx \quad = xe^x \Big|_1^2 = 2e^2 - 1 \cdot e^1 = 2e^2 - e$$

(a) $2e^2$

(b) $2e^2 - e$

(c) $xe^x$

(d) $3e^2 - 2e$

# Finding Area

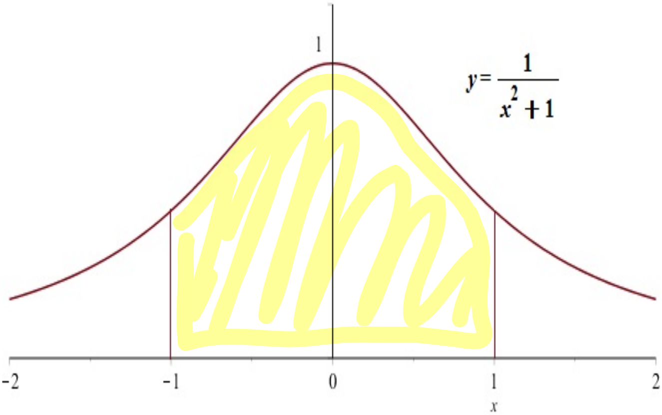


Figure: Use the FTC to find the area of the region under the graph of $y = \frac{1}{1+x^2}$, bounded below by the $x$-axis, and between the vertical lines $x = -1$ and $x = 1$.

Since $y$ is continuous and positive

$$\text{Area} = \int_{-1}^{1} \frac{1}{x^2+1}\,dx = \tan^{-1}x \Big|_{-1}^{1}$$

$$= \tan^{-1}1 - \tan^{-1}(-1)$$

$$= \frac{\pi}{4} - \left(\frac{-\pi}{4}\right)$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

# An Observation

If $f$ is differentiable on $[a, b]$, note that

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This says that:

The integral of the **rate of change** of $f$ over the interval $[a, b]$ is the **net change** of the function, $f(b) - f(a)$, over this interval.

# Example

Fluid is pumped into a reservoir at a rate of $R(t)$ gallons per minute. Consider the integral

$$\int_0^{10} R(t)\,dt = 2000$$

(a) What are the units of this integral?

R(t) is in gal/min     Change in volume

ie. $R(t) = V'(t)$ if $V(t)$ is the volume.

Here dt is in units of minutes

The integral, 2000, is in gallons.

# Example

Fluid is pumped into a reservoir at a rate of $R(t)$ gallons per minute. Consider the integral

$$\int_0^{10} R(t)\,dt = 2000$$

(b) Interpret this integral.

It's the net change in volume.

It's the amount of fluid added during that

10 minutes.

# Section 5.4: Properties of the Definite Integral

Suppose that $f$ and $g$ are integable on $[a, b]$ and let $k$ be constant.

I. $$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

II. $$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

II. $$\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

# The Sum/Difference in General

If $f_1, f_2, \ldots, f_n$ are integrable on $[a, b]$ and $k_1, k_2, \ldots, k_n$ are constants, then

$$\int_a^b [k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)]\, dx =$$

$$k_1 \int_a^b f_1(x)\, dx + k_2 \int_a^b f_2(x)\, dx + \cdots + k_n \int_a^b f_n(x)\, dx$$

# Example

Evaluate $\displaystyle\int_1^2 \frac{x^3+2x^2+4}{x}\,dx = \int_1^2 \left(\frac{x^3}{x} + \frac{2x^2}{x} + \frac{4}{x}\right) dx$

$$= \int_1^2 \left(x^2 + 2x + \frac{4}{x}\right) dx = \int_1^2 x^2\,dx + 2\int_1^2 x\,dx + 4\int_1^2 \frac{1}{x}\,dx$$

$$= \frac{x^3}{3}\Big|_1^2 + \cancel{2}\cdot\frac{x^2}{\cancel{2}}\Big|_1^2 + 4\ln|x|\,\Big|_1^2$$

$$= \frac{2^3}{3} - \frac{1^3}{3} + \left(2^2 - 1^2\right) + \left(4\ln|2| - 4\ln|1|\right)$$

$$= \frac{8}{3} - \frac{1}{3} + (4-1) + 4\ln 2 - 4\cdot 0 = \frac{7}{3} + 3 + 4\ln 2 = \frac{16}{3} + 4\ln 2$$

# Question

Evaluate $\int_0^1 (3x^2+4x-2)\,dx$ $= x^3 + 2x^2 - 2x \Big|_0^1$

$= 1^3 + 2\cdot 1^2 - 2\cdot 1 - (0+0-0)$

$= 1$

(a) 5

(b) $x^3 + 2x^2 - 2x$

(c) 1

(d) $6x + 4$

# Properties of Definite Integrals Continued...

If $f$ is integrable on any interval containing the numbers $a$, $b$, and $c$, then

(IV) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

To integrate from a to b, we can go from a to c and from c to b, and add the results.

# Example

Suppose $F'(x) = f(x)$ for all $x$. Show that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

By the FTC $\int_a^b f(x)dx = F(b) - F(a)$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \left(F(c) - F(a)\right) + \left(F(b) - F(c)\right)$$

$$= \cancel{F(c)} - F(a) + F(b) - \cancel{F(c)}$$

$$= F(b) - F(a) = \int_a^b f(x)\,dx$$

# Example

Suppose

$$\int_{-1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{2}^{4} f(x)\,dx = 3.$$

Evaluate $\displaystyle\int_{-1}^{2} f(x)\,dx$

Note $\displaystyle\int_{-1}^{4} f(x)\,dx = \int_{-1}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx$

$\Rightarrow \quad -2 = \displaystyle\int_{-1}^{2} f(x)\,dx + 3$

$\Rightarrow \quad \displaystyle\int_{-1}^{2} f(x)\,dx = -2 - 3 = -5$

# Properties: Bounds on Integrals

(V) If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

(VI) And, as an immediate consequence of (V), if $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

If $f$ is continuous on $[a, b]$, we can take $m$ to be the absolute minimum value and $M$ the absolute maximum value of $f$ as guaranteed by the Extreme Value Theorem.

# Example of Bounding

Not every integral can be evaluated in a straight forward way. For example $f(x) = e^{-x^2}$ does not have an elementary antiderivative* We can show that

$$\frac{1}{e} \leq e^{-x^2} \leq 1 \quad \text{for all } x \text{ in} \quad [0,1].$$

So even though evaluating $\int_0^1 e^{-x^2}\,dx$ is not *directly* possible, we can say that

$$\frac{1}{e} = \frac{1}{e}(1-0) \leq \int_0^1 e^{-x^2}\,dx \leq 1(1-0) = 1$$

*This function is a scaled version of the normal distribution function in probability and statistics.