

Section 11.2: Fourier Series

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad \text{and}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Fourier Series on an interval $(-p, p)$

The orthogonality relations provide for an expansion of a function f defined on $(-p, p)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \left(\frac{n\pi x}{p} \right) + b_n \sin \left(\frac{n\pi x}{p} \right) \right)$$

where

$$a_0 = \frac{1}{p} \int_{-p}^p f(x) dx,$$

$$a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \left(\frac{n\pi x}{p} \right) dx, \quad \text{and}$$

$$b_n = \frac{1}{\pi} \int_{-p}^p f(x) \sin \left(\frac{n\pi x}{p} \right) dx$$

An interesting observation...

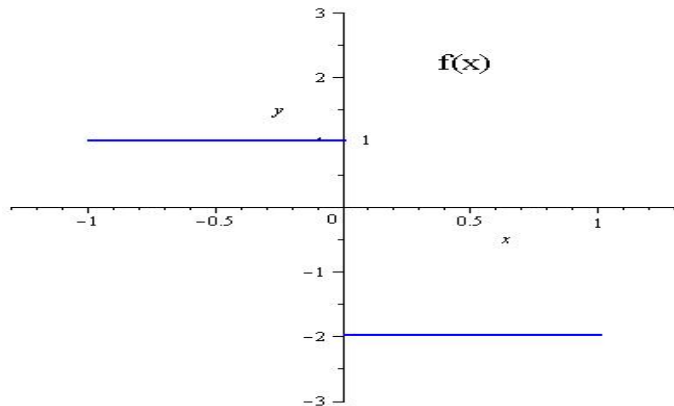
Note that the constant value is

$$\frac{a_0}{2} = \frac{1}{2p} \int_{-p}^p f(x) dx$$

This is the average value of f over
the interval $[-p, p]$ (or $(-p, p)$).

Example:

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \leq x < 1 \end{cases}$$



Example

We determined the Fourier series for this function is

$$f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

Note that f is piecewise continuous, but has a jump discontinuity at zero. Every term in the above sum however is continuous. A reasonable question here is whether the infinite sum is in fact continuous, and if so what is the connection between the series and the function at the discontinuity?

Convergence of the Series

Theorem: If f is continuous at x_0 in $(-p, p)$, then the series converges to $f(x_0)$ at that point. If f has a jump discontinuity at the point x_0 in $(-p, p)$, then the series **converges in the mean** to the average value

$$\frac{1}{2} \left(\lim_{x \rightarrow x_0^-} f(x) + \lim_{x \rightarrow x_0^+} f(x) \right)$$

at that point.

We can also note that it is possible to evaluate the series outside of the original interval. The series extends the original function into one that is $2p$ -periodic.

Find the Fourier Series for $f(x) = x$, $-1 < x < 1$

$$p=1$$

$$a_0 = \frac{1}{1} \int_{-1}^1 f(x) dx = \int_{-1}^1 x dx = \left. \frac{x^2}{2} \right|_{-1}^1 = \frac{1^2}{2} - \frac{(-1)^2}{2} = 0$$

$$a_n = \frac{1}{1} \int_{-1}^1 f(x) \cos\left(\frac{n\pi x}{1}\right) dx = \int_{-1}^1 x \cos(n\pi x) dx$$

$$= \left. \frac{x}{n\pi} \sin(n\pi x) \right|_{-1}^1 - \frac{1}{n\pi} \int_{-1}^1 \sin(n\pi x) dx$$

$$= \frac{1}{n\pi} \sin(n\pi) + \frac{1}{n\pi} \sin(-n\pi) + \frac{1}{(n\pi)^2} \cos(n\pi x) \Big|_{-1}^1$$

0'' 0''

By parts

$$u = x \quad du = dx$$

$$dv = \cos(n\pi x) dx$$

$$v = \frac{1}{n\pi} \sin(n\pi x)$$

$$= \frac{1}{(n\pi)^2} \cos(n\pi) - \frac{1}{(n\pi)^2} \cos(-n\pi) = 0$$

$$b_n = \frac{1}{1} \int_{-1}^1 f(x) \sin\left(\frac{n\pi x}{1}\right) dx = \int_{-1}^1 x \sin(n\pi x) dx$$

$$= \left. \frac{-x}{n\pi} \cos(n\pi x) \right|_{-1}^1 + \frac{1}{n\pi} \int_{-1}^1 \cos(n\pi x) dx$$

$$= \left. \frac{-1}{n\pi} \cos(n\pi) - \frac{1}{n\pi} \cos(-n\pi) + \left(\frac{1}{n\pi}\right)^2 \sin(n\pi x) \right|_{-1}^1$$

By parts

$$u = x \quad du = dx$$

$$dv = \sin(n\pi x) dx$$

$$v = \frac{-1}{n\pi} \cos(n\pi x)$$

$$= \frac{-2(-1)^n}{n\pi} + \frac{1}{(n\pi)^2} \sin(n\pi) - \frac{1}{(n\pi)^2} \sin(-n\pi)$$

$$b_n = \frac{2(-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

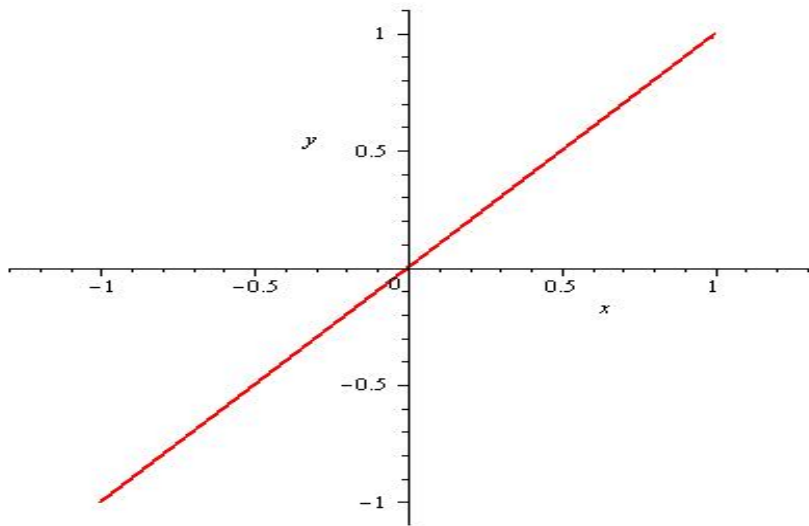


Figure: Plot of $f(x) = x$ for $-1 < x < 1$

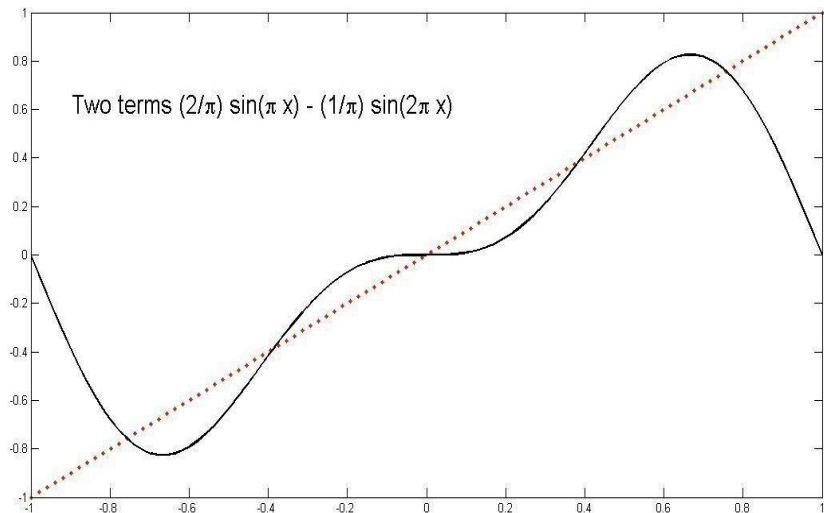


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with two terms of the Fourier series.

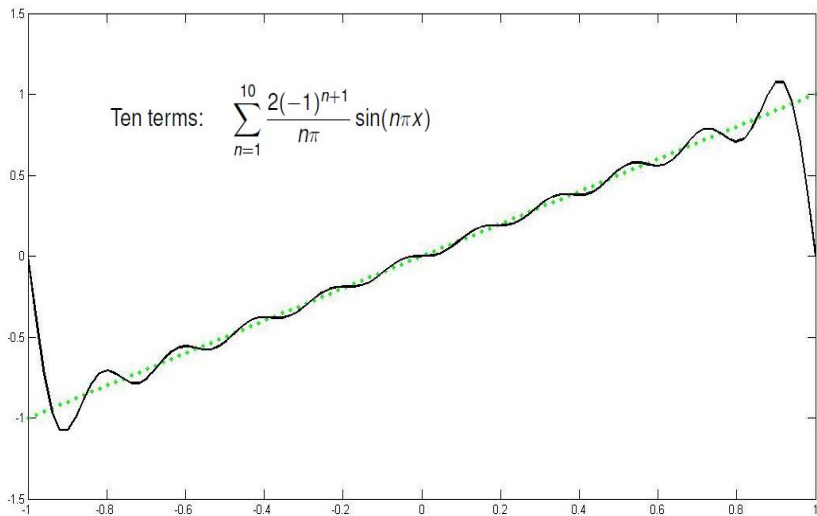


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with 10 terms of the Fourier series

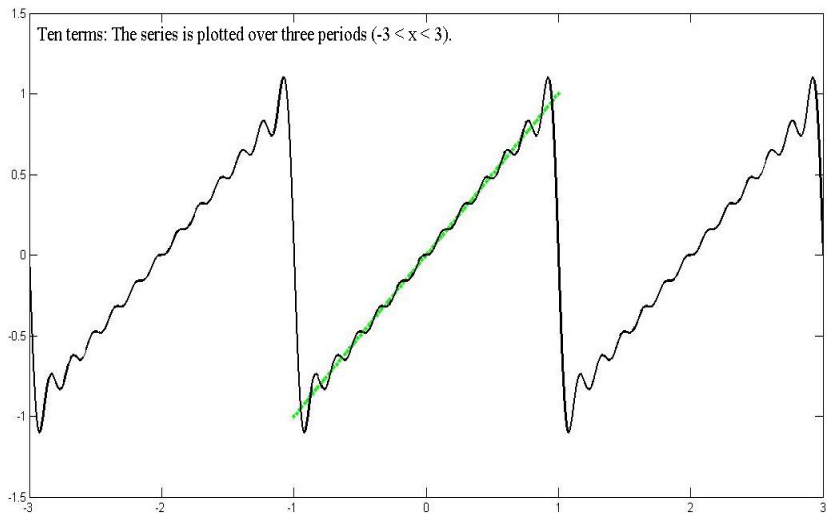


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with the Fourier series plotted on $(-3, 3)$

Symmetry

Suppose f is defined on an interval containing x and $-x$.

If $f(-x) = f(x)$ for all x , then f is said to be **even**.

If $f(-x) = -f(x)$ for all x , then f is said to be **odd**.

For example, $f(x) = x^n$ is even if n is even and is odd if n is odd. The trigonometric function $g(x) = \cos x$ is even, and $h(x) = \sin x$ is odd.

Integrals on symmetric intervals

If f is an even function on $(-p, p)$, then

$$\int_{-p}^p f(x) dx = 2 \int_0^p f(x) dx.$$

If f is an odd function on $(-p, p)$, then

$$\int_{-p}^p f(x) dx = 0.$$

Products of Even and Odd functions

$$\text{Even} \times \text{Even} = \text{Even},$$

and

$$\text{Odd} \times \text{Odd} = \text{Even}.$$

While

$$\text{Even} \times \text{Odd} = \text{Odd}.$$

So, suppose f **is even** on $(-p, p)$. This tells us that $f(x) \cos(nx)$ is **even** for all n and $f(x) \sin(nx)$ is **odd** for all n .

And, if f **is odd** on $(-p, p)$. This tells us that $f(x) \sin(nx)$ is **even** for all n and $f(x) \cos(nx)$ is **odd** for all n .

Fourier Series of an Even Function

If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{2}{p} \int_0^p f(x) dx$$

and

$$a_n = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{n\pi x}{p}\right) dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p, p)$, then the Fourier series of f has only sine terms. Moreover

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

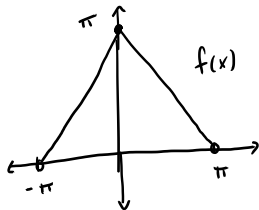
$$b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right) dx.$$

Find the Fourier series of f

Here, $p = \pi$

$$f(x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ \pi - x, & 0 \leq x < \pi \end{cases}$$

Plot to check for symmetry:



f is clearly even (no sine terms)

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{2}{\pi} \left[\pi x - \frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\pi^2 - \frac{\pi^2}{2} - 0 \right] = \frac{2}{\pi} \left(\frac{\pi^2}{2} \right) = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} (\pi-x) \cos(nx) dx$$

$$= \frac{2}{\pi} \left[\frac{(\pi-x)}{n} \sin(nx) \right]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \sin(nx) dx$$

$$= \frac{2}{\pi} \left[-\frac{1}{n^2} \cos(nx) \right]_0^{\pi}$$

$$= \frac{-2}{\pi n^2} (\cos(n\pi) - \cos 0)$$

$$= \frac{-2}{\pi n^2} ((-1)^n - 1) = \frac{2}{\pi n^2} (1 - (-1)^n)$$

By parts

$$u = \pi - x$$

$$du = -dx$$

$$dv = \cos(nx) dx$$

$$v = \frac{1}{n} \sin(nx)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(1 - (-1)^n)}{\pi n^2} \cos(nx)$$

Half Range Sine and Half Range Cosine Series

Suppose f is only defined for $0 < x < p$. We can **extend** f to the left, to the interval $(-p, 0)$, as either an even function, or as an odd function. Then we can express f with **two distinct** series:

$$\text{Half range cosine series} \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

$$\text{where} \quad a_0 = \frac{2}{p} \int_0^p f(x) dx \quad \text{and} \quad a_n = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{n\pi x}{p}\right) dx.$$

$$\text{Half range sine series} \quad f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

$$\text{where} \quad b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right) dx.$$

Extending a Function to be Odd

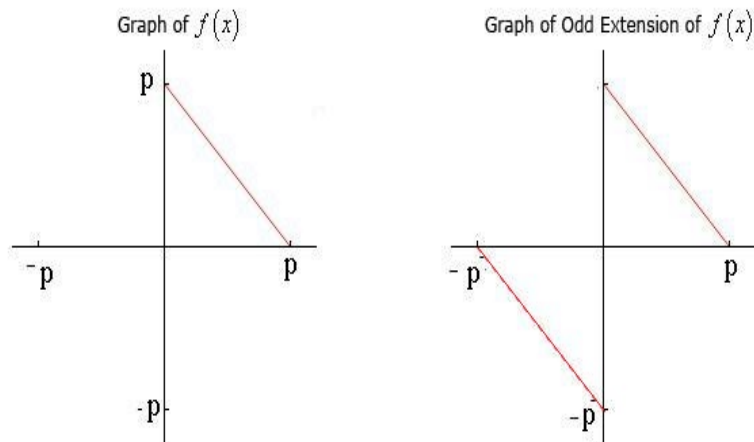


Figure: $f(x) = p - x$, $0 < x < p$ together with its **odd** extension.

Extending a Function to be Even

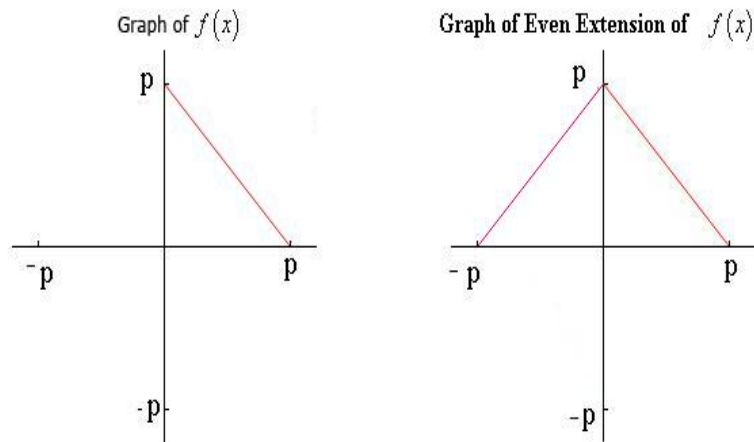


Figure: $f(x) = p - x$, $0 < x < p$ together with its **even** extension.