November 18 Math 2306 sec 54 Fall 2015

Section 11.2: Fourier Series

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

Where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}$$

Fourier Series on an interval $(-p, p)$

The set of functions The set $\left\{1, \cos\left(\frac{n\pi x}{p}\right), \sin\left(\frac{m\pi x}{p}\right) | n, m \geq 1\right\}$ is orthogonal on $(-p, p)$. Moreover, we have the properties

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\, dx = 0 \quad \text{and} \quad \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \;\; \text{for all} \;\; n, m \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \quad \text{for all} \quad m, n \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \cos\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases},$$

$$\int_{-p}^{p} \sin\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases}.$$

Fourier Series on an interval $(-p, p)$

The orthogonality relations provide for an expansion of a function f defined on $(-p, p)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{p} \int_{-p}^{p} f(x)\, dx, \\ a_n &= \frac{1}{p} \int_{-p}^{p} f(x) \cos\left(\frac{n\pi x}{p}\right) dx, \quad \text{and} \\ b_n &= \frac{1}{p} \int_{-p}^{p} f(x) \sin\left(\frac{n\pi x}{p}\right) dx \end{aligned}$$

An interesting observation...

Note that the constant value is

$$\frac{a_0}{2} = \frac{1}{2p} \int_{-p}^{p} f(x)\, dx$$

The integral of f over [-p,p] divided by

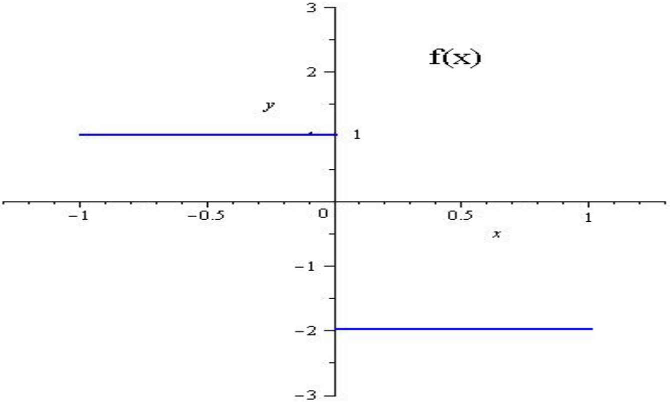
$2p = p-(-p)$ (length of the interval).

$\frac{a_0}{2}$ is the average value of f on [-p,p].

Find the Fourier series of f

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}$$

The interval $(-p, p)$ is $(-1, 1)$ here, so $p = \underline{1}$.



$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} dx + \int_{0}^{1}(-2)\,dx$$

$$= x\Big|_{-1}^{0} + \left[-2x\Big|_{0}^{1}\right] = 0-(-1) + [-2-0] = -1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)dx = \int_{-1}^{0}\cos(n\pi x)\,dx + \int_{0}^{1}(-2)\cos(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} - \frac{2}{n\pi}\sin(n\pi x)\Big|_{0}^{1}$$

$$= \frac{1}{n\pi}\sin(0) - \frac{1}{n\pi}\sin(-n\pi) - \left[\frac{2}{n\pi}\sin(n\pi) - \frac{2}{n\pi}\sin 0\right] = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\, \sin\left(\frac{n\pi x}{1}\right) dx$$

$$= \int_{-1}^{0} \sin(n\pi x)\, dx + \int_{0}^{1} (-2) \sin(n\pi x)\, dx$$

$$= \frac{-1}{n\pi} \cos(n\pi x)\Big|_{-1}^{0} + \frac{2}{n\pi} \cos(n\pi x)\Big|_{0}^{1}$$

$$= \frac{-1}{n\pi}\left[\cos 0 - \cos(-n\pi)\right] + \frac{2}{n\pi}\left[\cos(n\pi) - \cos 0\right]$$

$$= \frac{-1}{n\pi} + \frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n\pi} - \frac{2}{n\pi} = \frac{3}{n\pi}\left((-1)^n - 1\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

$$f(x) = \frac{-1}{2} + \sum_{n=1}^{\infty} \frac{3}{n\pi}\left((-1)^n - 1\right) \sin(n\pi x)$$

Q: f is discontinuous @ $x=0$.

Is the sum continuous @ $x=0$?

What is the connection between f and the series?

Convergence of the Series

Theorem: If f is continuous at x_0 in $(-p, p)$, then the series converges to $f(x_0)$ at that point. If f has a jump discontinuity at the point x_0 in $(-p, p)$, then the series **converges in the mean** to the average value

$$\frac{1}{2}\left(\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right)$$

at that point.

In addition, the series extends the function $f(x)$ to be $2p$-periodic.

Find the Fourier Series for $f(x) = x$, $-1 < x < 1$

Here $p = 1$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} x\,dx = \left.\frac{x^2}{2}\right|_{-1}^{1} = \frac{1^2}{2} - \frac{(-1)^2}{2} = 0$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)dx = \int_{-1}^{1} x\cos(n\pi x)\,dx$$

By parts

$u = x \quad du = dx$

$dv = \cos(n\pi x)\,dx$

$v = \frac{1}{n\pi}\sin(n\pi x)$

$$= \left.\frac{x}{n\pi}\sin(n\pi x)\right|_{-1}^{1} - \frac{1}{n\pi}\int_{-1}^{1}\sin(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\sin(n\pi) - \frac{-1}{n\pi}\sin(-n\pi) + \left.\frac{1}{(n\pi)^2}\cos(n\pi x)\right|_{-1}^{1}$$

$$= \frac{1}{(n\pi)^2}\left[\cos(n\pi) - \cos(-n\pi)\right] = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right)dx = \int_{-1}^{1} x\sin(n\pi x)\,dx$$

By parts

$u = x \quad du = dx$

$dv = \sin(n\pi x)dx$

$v = \frac{-1}{n\pi}\cos(n\pi x)$

$$= \frac{-x}{n\pi}\cos(n\pi x)\Big|_{-1}^{1} + \frac{1}{n\pi}\int_{-1}^{1}\cos(n\pi x)\,dx$$

$$= \frac{-1}{n\pi}\cos(n\pi) - \frac{-(-1)}{n\pi}\cos(-n\pi) + \frac{1}{(n\pi)^2}\sin(n\pi x)\Big|_{-1}^{1}$$

$$= \frac{-2}{n\pi}(-1)^n + \frac{1}{(n\pi)^2}\Big(\sin(n\pi) - \underbrace{\sin(-n\pi)}_{0}\Big)$$

$$b_n = \frac{2}{n\pi}(-1)^{n+1} \qquad \text{recall} \quad a_0 = 0 \quad a_n = 0 \quad n = 1, 2, \ldots$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

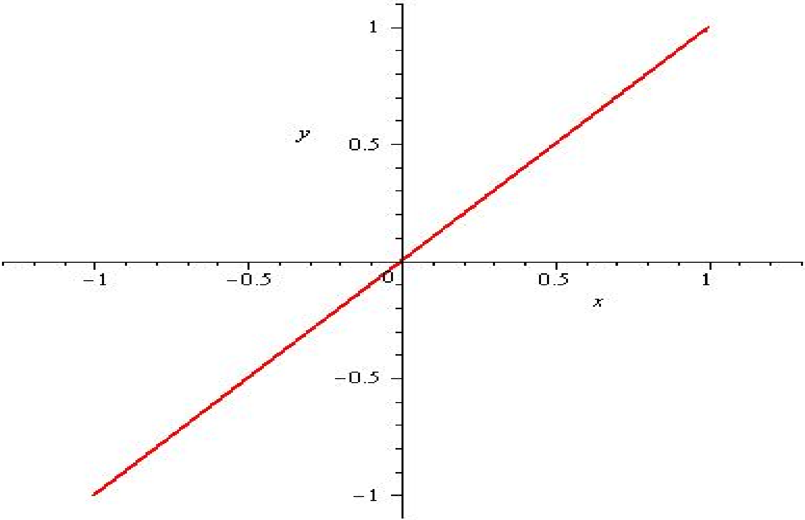


Figure: Plot of $f(x) = x$ for $-1 < x < 1$

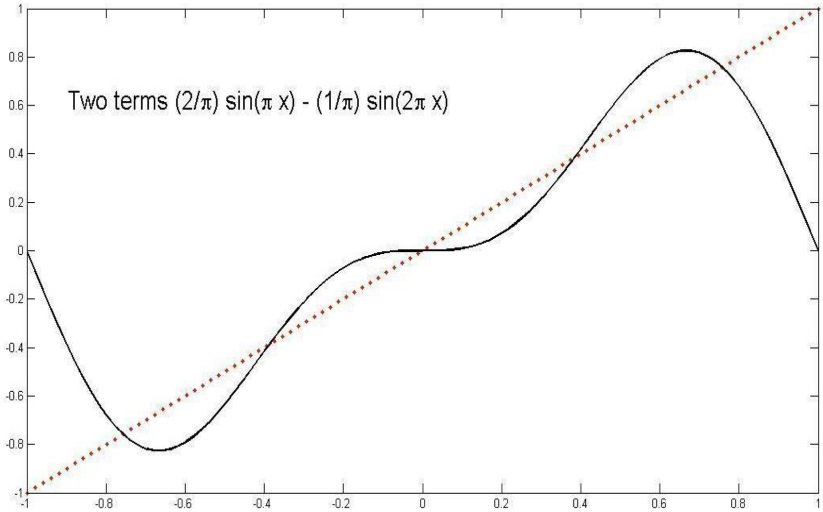


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with two terms of the Fourier series.

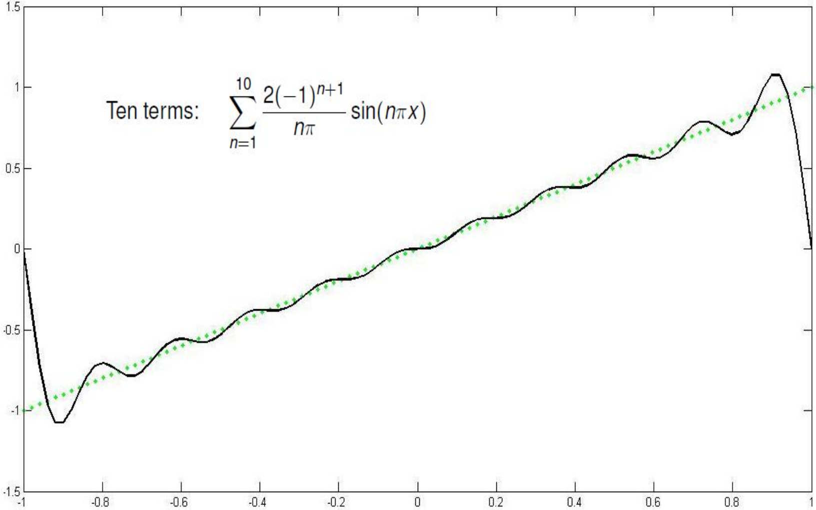


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with 10 terms of the Fourier series

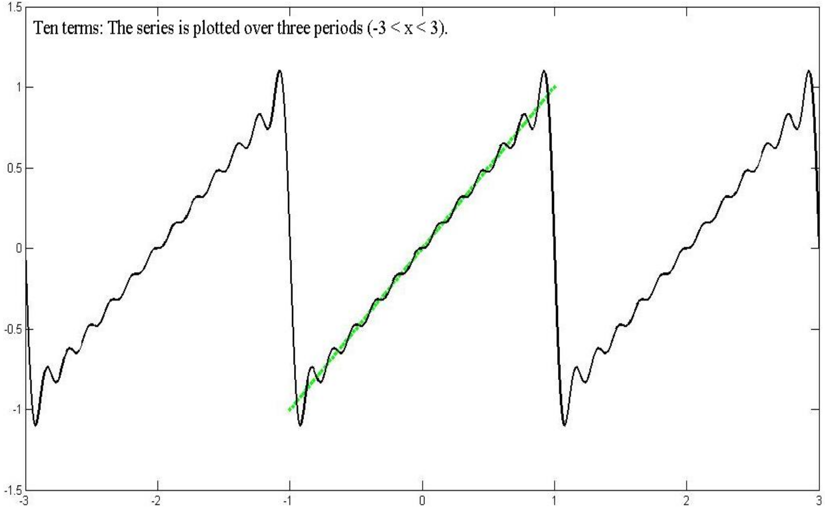


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with the Fourier series plotted on $(-3, 3)$

Symmetry

Suppose f is defined on an interval containing x and $-x$.

If $f(-x) = f(x)$ for all x, then f is said to be **even.**

If $f(-x) = -f(x)$ for all x, then f is said to be **odd.**

For example, $f(x) = x^n$ is even if n is even and is odd if n is odd. The trigonometric function $g(x) = \cos x$ is even, and $h(x) = \sin x$ is odd.

Integrals on symmetric intervals

If f is an even function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 2\int_{0}^{p} f(x)\,dx.$$

If f is an odd function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 0.$$

Products of Even and Odd functions

$$\text{Even} \quad \times \quad \text{Even} \quad = \quad \text{Even},$$

and

$$\text{Odd} \quad \times \quad \text{Odd} \quad = \quad \text{Even}.$$

While

$$\text{Even} \quad \times \quad \text{Odd} \quad = \quad \text{Odd}.$$

So, suppose f **is even** on $(-p, p)$. This tells us that $f(x)\cos(nx)$ is even for all n and $f(x)\sin(nx)$ is odd for all n.

And, if f **is odd** on $(-p, p)$. This tells us that $f(x)\sin(nx)$ is even for all n and $f(x)\cos(nx)$ is odd for all n

Fourier Series of an Even Function

If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

and

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p, p)$, then the Fourier series of f has only sine terms. Moreover

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

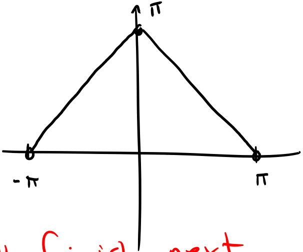
$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Find the Fourier series of f

$$f(x) = \begin{cases} x+\pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

We can check for symmetry.

Plot of f



We'll finish next time.

f is even

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$