November 20 Math 2306 sec 51 Fall 2015

Section 11.3: Fourier Cosine and Sine Series

If f is even on $(-p,p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{p}\right)$$

where

$$a_0=\frac{2}{p}\int_0^p f(x)\,dx$$

and

$$a_n=\frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p, p)$, then the Fourier series of f has only sine terms. Moreover

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Half Range Sine and Half Range Cosine Series

Suppose f is only defined for $0 < x < p$. We can **extend** f to the left, to the interval $(-p, 0)$, as either an even function, or as an odd function. Then we can express f with **two distinct** series:

Half range cosine series $\quad f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$

where $\quad a_0 = \dfrac{2}{p}\displaystyle\int_0^p f(x)\,dx \quad$ and $\quad a_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$

Half range sine series $\quad f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$

where $\quad b_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$

Find the Half Range Sine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

here $p = 2$

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

By parts

$u = 2-x, \quad du = -dx$

$dv = \sin\left(\frac{n\pi x}{2}\right)dx$

$v = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$

$$= \frac{-2}{n\pi}(2-x)\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 - \frac{2}{n\pi}\underbrace{\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx}_{0}$$

$$= \frac{-2}{n\pi}\left[(2-2)\cos(n\pi) - 2\cos 0\right] = \frac{4}{n\pi}$$

$$\boxed{\begin{array}{l}\text{The half range sine series is} \\[1em] \displaystyle f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\end{array}}$$

Find the Half Range Cosine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (2-x)\,dx = 2x - \frac{x^2}{2}\Big]_0^2 = 4 - \frac{4}{2} = 2$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \underbrace{\frac{2}{n\pi}(2-x)\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2}_{0} + \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx$$

$u = 2 - x \qquad du = -dx$

$dv = \cos\left(\frac{n\pi x}{2}\right)dx$

$v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$

$$= -\left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2$$

$$= \frac{-4}{n^2\pi^2}\left[\cos(n\pi) - \cos 0\right]$$

$$= \frac{4}{n^2\pi^2}\left(1 - (-1)^n\right)$$

The half range cosine series is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)\cos\left(\frac{n\pi x}{2}\right).$$

Plots of f with Half range series

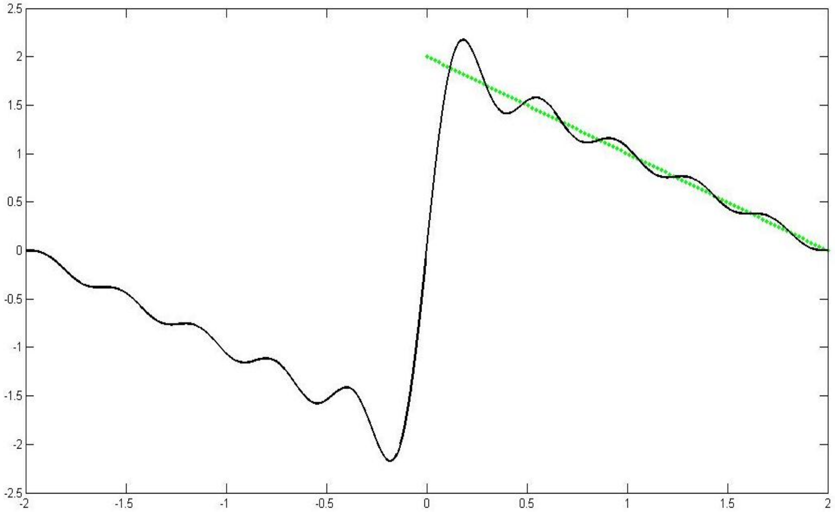

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series.

Plots of f with Half range series

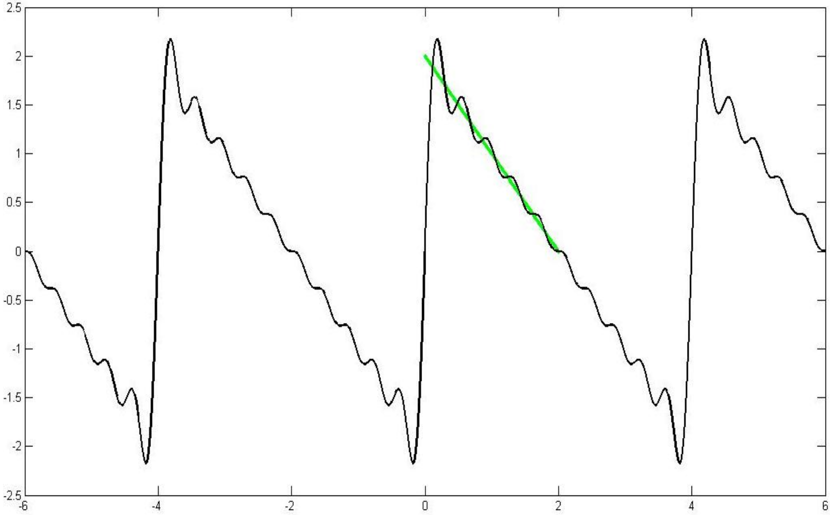

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series, and the series plotted over $(-6, 6)$

Plots of f with Half range series

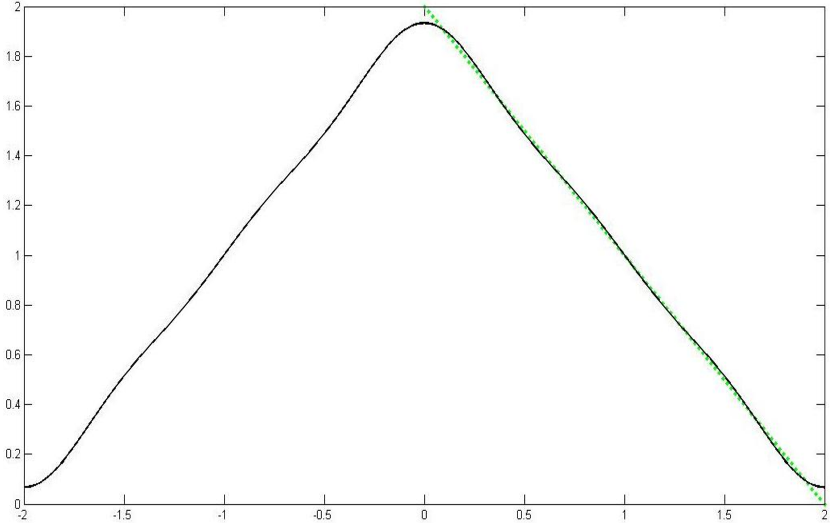

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series.

Plots of f with Half range series

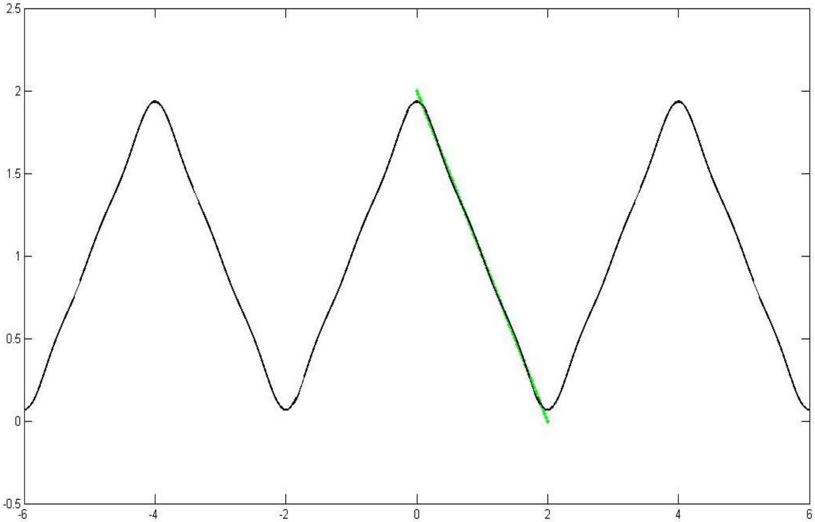

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series, and the series plotted over $(-6, 6)$