November 20 Math 2306 sec 54 Fall 2015

Section 11.3: Fourier Cosine and Sine Series

If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

and

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p, p)$, then the Fourier series of f has only sine terms. Moreover

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Find the Fourier series of f

$$f(x) = \begin{cases} x+\pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

We graphed f and determined that it is even. So the series will only contain constant and cosine terms

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{with}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\,dx = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right] = \frac{2}{\pi}\left(\frac{\pi^2}{2}\right) = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[\underbrace{\frac{\pi - x}{n}\sin(nx)\Big|_0^{\pi}}_{0} + \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right.$$

$u = \pi - x \quad du = -dx$

$dv = \cos(nx)\,dx$

$v = \frac{1}{n}\sin(nx)$

$$= \frac{2}{\pi}\left[\frac{-1}{n^2}\cos(nx)\Big|_0^{\pi}\right] = \frac{-2}{n^2\pi}\left[\cos(n\pi) - \cos 0\right]$$

$$= \frac{2}{n^2\pi}\left(1-(-1)^n\right)$$

The Fourier Series is

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left(1-(-1)^n\right)\cos(nx).$$

Half Range Sine and Half Range Cosine Series

Suppose f is only defined for $0 < x < p$. We can **extend** f to the left, to the interval $(-p, 0)$, as either an even function, or as an odd function. Then we can express f with **two distinct** series:

Half range cosine series $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$

where $\quad a_0 = \frac{2}{p}\int_0^p f(x)\,dx \quad$ and $\quad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$

Half range sine series $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$

where $\quad b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$

Extending a Function to be Odd

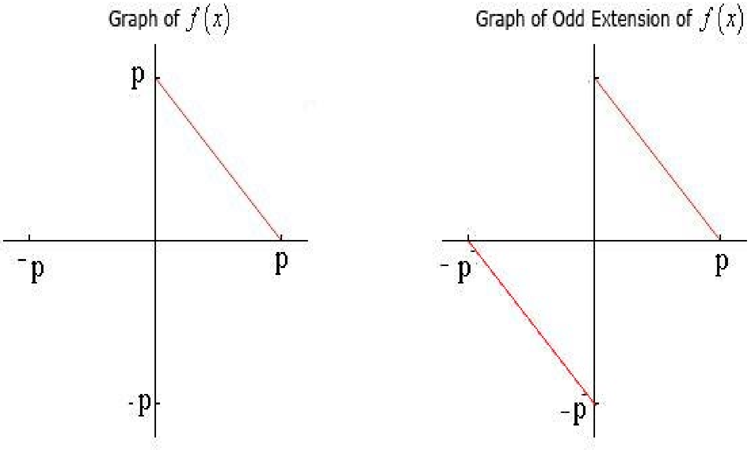

Figure: $f(x) = p - x$, $0 < x < p$ together with its **odd** extension.

Extending a Function to be Even

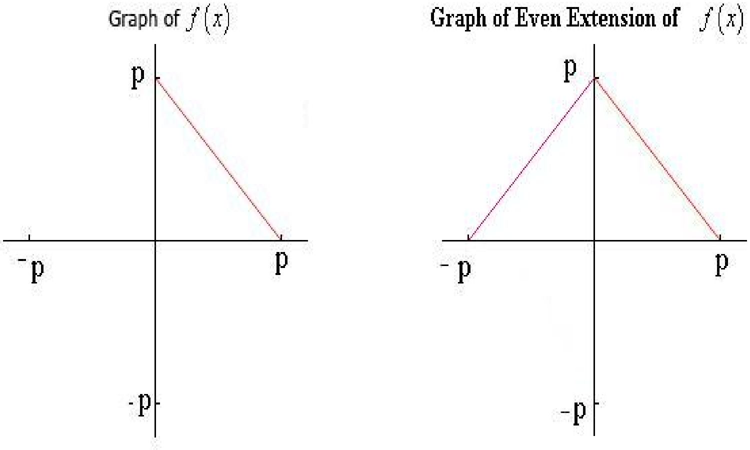

Figure: $f(x) = p - x$, $0 < x < p$ together with its **even** extension.

Find the Half Range Sine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$p = 2$

$$b_n = \frac{2}{2}\int_0^2 f(x)\,\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 (2-x)\,\sin\left(\frac{n\pi x}{2}\right)dx$$

$u = 2 - x, \quad du = -dx$

$dv = \sin\left(\frac{n\pi x}{2}\right)dx$

$$= \frac{-2(2-x)}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 - \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$v = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$

$$= \frac{-2\cdot 0}{n\pi}\cos(n\pi) - \frac{-2\cdot 2}{n\pi}\cos(0) - \frac{2^2}{n^2\pi^2}\underbrace{\cancel{\sin\left(\frac{n\pi x}{2}\right)}}_{0}\Bigg|_0^2$$

$$= \frac{4}{n\pi}$$

The half range Sine Series is

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$

Find the Half Range Cosine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (2-x)\,dx = 2x - \frac{x^2}{2}\Big|_0^2 = 4 - \frac{4}{2} = 2$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$u = 2 - x \qquad du = -dx$$

$$dv = \cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$

$$= \underbrace{\frac{2(2-x)}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2}_{0} + \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{-2^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2$$

$$= \frac{-4}{n^2\pi^2}\left[\cos(n\pi) - \cos 0\right] = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

The half range cosine series is

$$\boxed{f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)\cos\left(\frac{n\pi x}{2}\right)}$$

Plots of f with Half range series

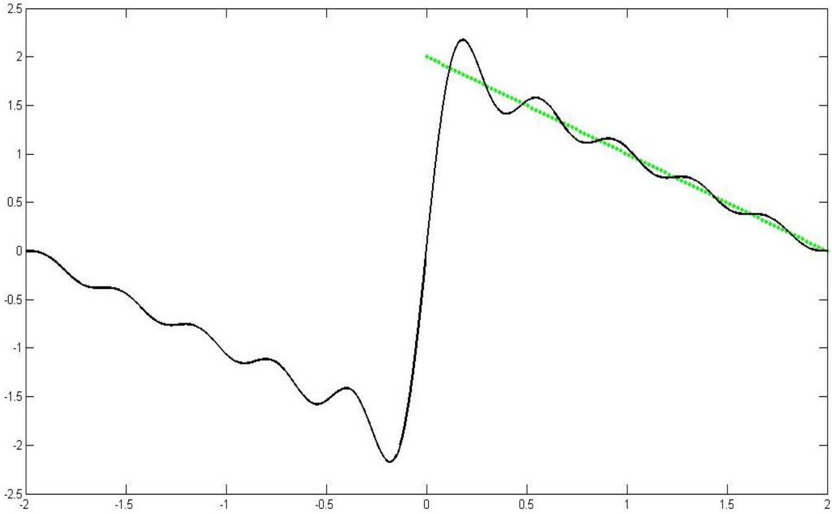

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series.

Plots of f with Half range series

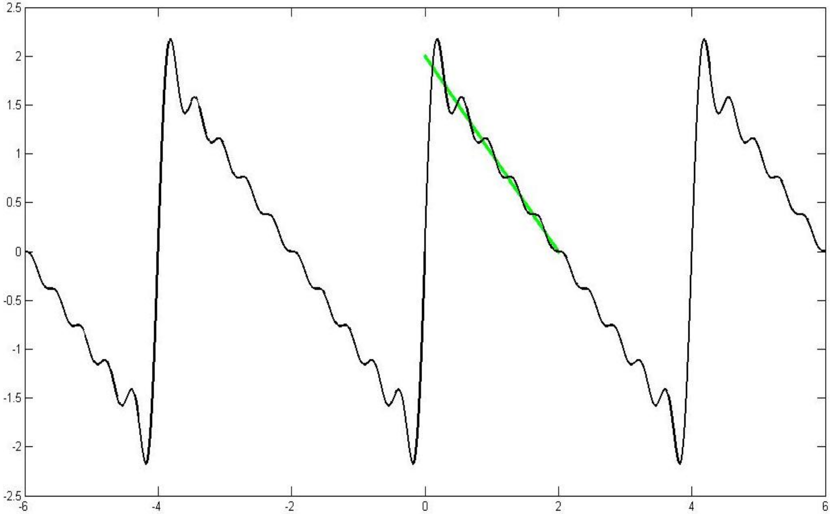

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series, and the series plotted over $(-6, 6)$

Plots of f with Half range series

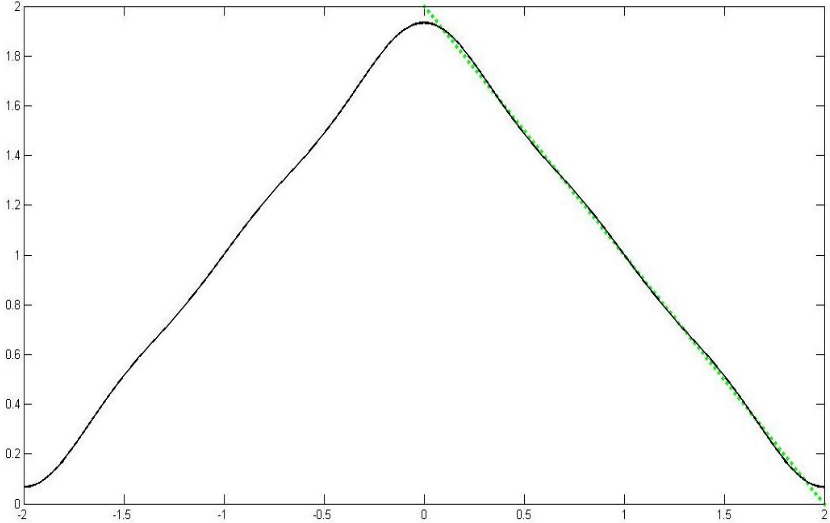

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series.

Plots of f with Half range series

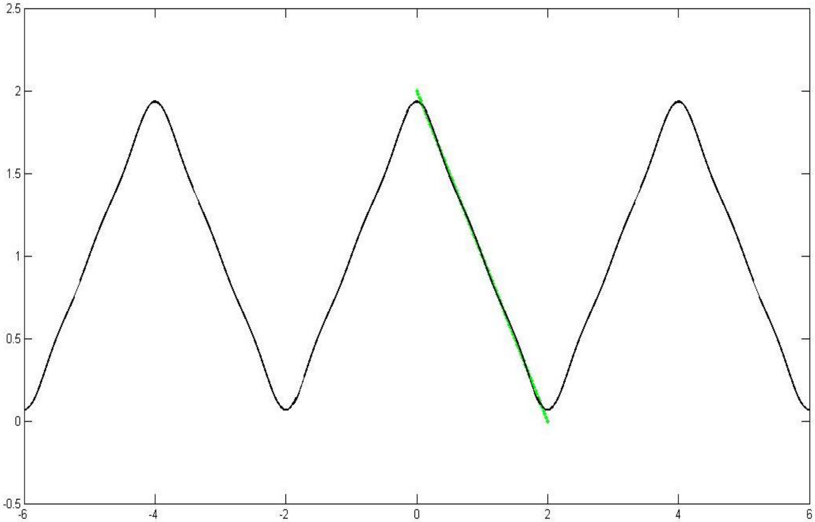

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series, and the series plotted over $(-6, 6)$

Half Range Series

For the given function, plot the graph of the function along with three full periods on the interval $(-3p, 3p)$ of (a) the half range cosine series and (b) the half range sine series.

$$f(x) = \begin{cases} x, & 0 \le x < \frac{3}{2} \\ 3 - x, & \frac{3}{2} \le x < 3 \end{cases}$$

here $p = 3$

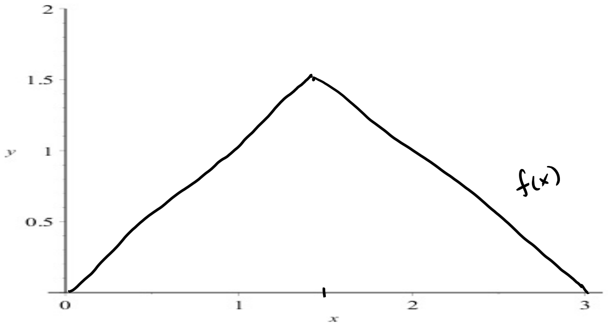

Figure: Plot of f alone.

(a) Even Extension

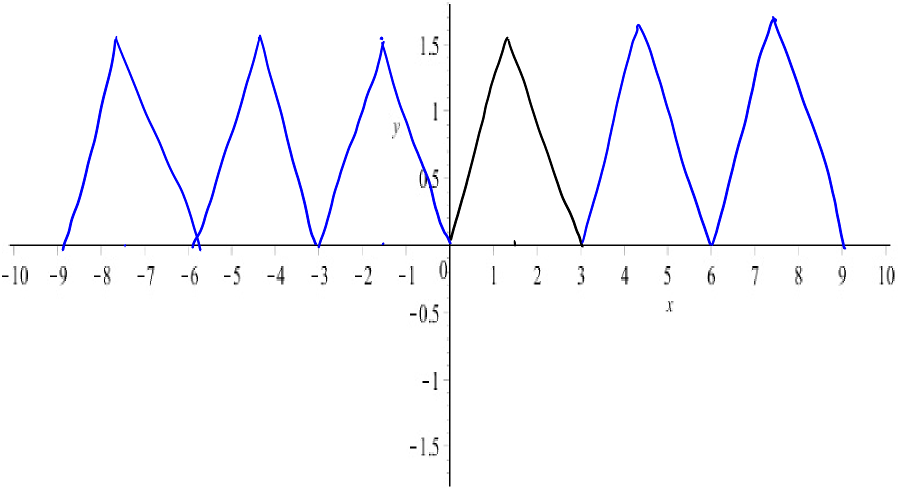

Figure: Half range cosine series.

(b) Odd Extension

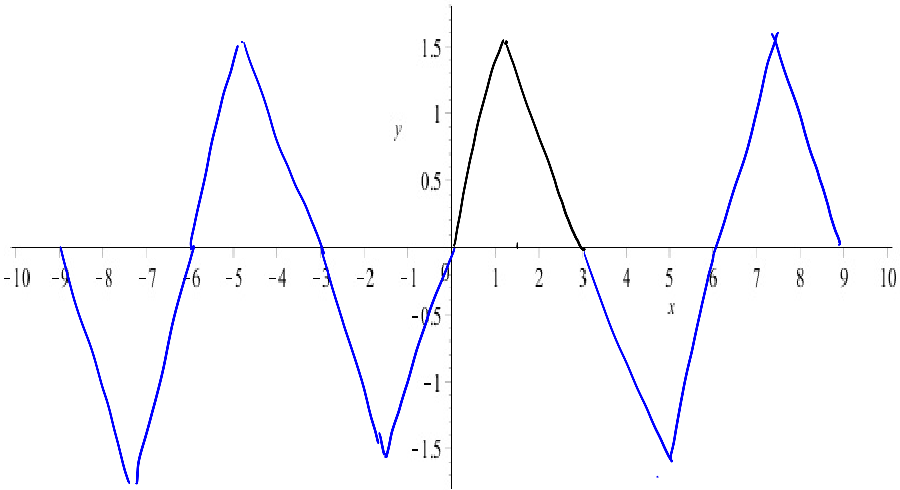

Figure: Half range sine series.