November 26 MATH 1113 sec. 51 Fall 2018

Section 7.2: Double & Half Angle IDs

Use $\sin(u+v) = \sin u \cos v + \sin v \cos u$ to obtain a formula for

$\sin(2u)$

$2u = u+u$

$$\sin(2u) = \sin(u+u)$$

$$= \sin u \cos u + \sin u \cos u$$

$$= 2 \sin u \cos u$$

Double Angle Formulas for the Cosine

Use $\cos(u+v) = \cos u \cos v - \sin u \sin v$ and $\cos^2 u + \sin^2 u = 1$ to find three formulas for

$\cos(2u)$: $\cos(u+u) = \cos u \cos u - \sin u \sin u$

$= \cos^2 u - \sin^2 u$

Since $\sin^2 u = 1 - \cos^2 u$

$\cos(2u) = \cos^2 u - (1 - \cos^2 u)$

$= 2\cos^2 u - 1$

$\sin u \qquad \cos^2 u = 1 - \sin^2 u$

$$\cos(2u) = 1 - \sin^2 u - \sin^2 u$$

$$= 1 - 2\sin^2 u$$

Question: Double Angle Formulas for the Tangent

From the sum formula $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$, it follows that

$\tan(2u) =$

(a) $\dfrac{2\tan u}{1 - 2\tan u}$

(b) $\dfrac{2\tan u}{1 - \tan^2 u}$

(c) $\dfrac{\tan^2 u}{1 - 2\tan u}$

(d) $\dfrac{\tan^2 u}{1 - \tan^2 u}$

Double Angle Formulas

$$\sin(2u) = 2\sin u \cos u$$

$$\begin{aligned}\cos(2u) &= \cos^2 u - \sin^2 u\\ &= 2\cos^2 u - 1\\ &= 1 - 2\sin^2 u\end{aligned}$$

$$\tan(2u) = \frac{2\tan u}{1-\tan^2 u}$$

Example

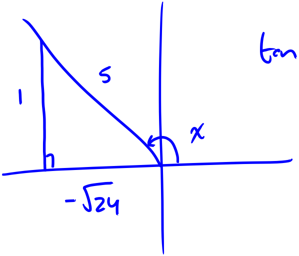
Suppose $\csc(x) = 5$ and $\cot(x) < 0$. Find the exact value of

(a) $\tan(2x)$: $\dfrac{2\tan x}{1-\tan^2 x}$

$\csc(x) > 0 \quad \cot(x) < 0$

x - has terminal side in quad II

(b) $\sec(2x)$



$\tan x = \dfrac{-1}{\sqrt{24}}$

a) $\tan(2x) = \dfrac{2\left(\frac{-1}{\sqrt{24}}\right)}{1-\left(\frac{-1}{\sqrt{24}}\right)^2}$

$= \dfrac{\frac{-2}{\sqrt{24}}}{1-\frac{1}{24}} \left(\dfrac{24}{24}\right)$

$$= \frac{-2\sqrt{24}}{24-1} = \frac{-2\sqrt{24}}{23}$$

b) $\sec(2x)$

From $\csc x = 5$, $\sin x = \frac{1}{5}$

so

$$\cos(2x) = 1 - 2\sin^2 x$$

$$= 1 - 2\left(\frac{1}{5}\right)^2 = 1 - \frac{2}{25}$$

$$= \frac{25-2}{25} = \frac{23}{25}$$

So

$$\sec(2x) = \frac{25}{23}$$

Half Angle IDs

Use the fact that $\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ and that $\frac{\pi}{4}=2\frac{\pi}{8}$ to find the exact value of

$\sin^2\left(\frac{\pi}{8}\right)$ $=\dfrac{\sqrt{2}-1}{2\sqrt{2}}$

$$\cos\left(\frac{\pi}{4}\right)=\cos\left(2\frac{\pi}{8}\right)=1-2\sin^2\left(\frac{\pi}{8}\right)$$

$$2\sin^2\left(\frac{\pi}{8}\right)=1-\cos\left(\frac{\pi}{4}\right)$$

$$\sin^2\left(\frac{\pi}{8}\right)=\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}$$

$$=\frac{1-\frac{1}{\sqrt{2}}}{2}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)=\frac{\sqrt{2}-1}{2\sqrt{2}}$$

Half Angle IDs

$$
\begin{aligned}
\sin^2 x &= \frac{1-\cos(2x)}{2} & \sin\left(\frac{x}{2}\right) &= \pm\sqrt{\frac{1-\cos x}{2}} \\
\cos^2 x &= \frac{1+\cos(2x)}{2} & \cos\left(\frac{x}{2}\right) &= \pm\sqrt{\frac{1+\cos x}{2}} \\
\tan^2 x &= \frac{1-\cos(2x)}{1+\cos(2x)} & \tan\left(\frac{x}{2}\right) &= \pm\sqrt{\frac{1-\cos(x)}{1+\cos(x)}}
\end{aligned}
$$

For a given value of x, only one of the signs $+$ or $-$ will apply. To choose the correct sign, determine which quadrant the angle $\frac{x}{2}$ is in when in standard position.

Determine the exact value of

(a) $\cos(22.5^\circ) = \pm\sqrt{\dfrac{1+\cos(2\cdot 22.5^\circ)}{2}}$

$2(22.5^\circ) = 45^\circ$

$0^\circ < 22.5^\circ < 90^\circ$

so

$\cos(22.5^\circ) > 0$

$$\cos(22.5^\circ) = \sqrt{\frac{1+\cos(45^\circ)}{2}}$$

$$= \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}\cdot\left(\frac{\sqrt{2}}{\sqrt{2}}\right)}$$

$$= \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

Question

In standard position, the angle $\frac{13\pi}{12}$ would have its terminal side in

(a) Quadrant 1

(b) Quadrant 2

$\frac{13\pi}{12} = \pi + \frac{\pi}{12}$

(c) Quadrant 3

(d) Quadrant 4

(e) $\frac{13\pi}{12}$ isn't a number, Dr. Ritter made it up

Question

The angle $\frac{13\pi}{6}$ is coterminal with

(a) $\frac{\pi}{6}$

$$\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$$

(b) $\frac{5\pi}{6}$

(c) $\frac{7\pi}{6}$

(d) $\frac{11\pi}{6}$

(e) $\frac{13\pi}{6}$ isn't a number, Dr. Ritter made it up

Question

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$

The exact value of $\sin\left(\dfrac{13\pi}{12}\right) = -\sqrt{\dfrac{1-\cos\left(\frac{13\pi}{6}\right)}{2}}$

$$= -\sqrt{\frac{1-\cos\left(\frac{\pi}{6}\right)}{2}}$$

$$= -\sqrt{\frac{1-\sqrt{3}/2}{2}}$$

$$= -\sqrt{\frac{2-\sqrt{3}}{4}} = -\frac{\sqrt{2-\sqrt{3}}}{2}$$

(a) $\dfrac{1}{4}$

(b) $-\dfrac{\sqrt{3}}{4}$

(c) $-\dfrac{\sqrt{2-\sqrt{3}}}{2}$

(d) $\dfrac{\sqrt{2-\sqrt{3}}}{2}$

Find the exact value

(c) $\tan\left(\frac{\theta}{2}\right)$ given $\cos\theta = \frac{4}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

From $\frac{3\pi}{2} < \theta < 2\pi$

$\frac{1}{2}\left(\frac{3\pi}{2}\right) < \frac{1}{2}\theta < \frac{1}{2}(2\pi)$

$\frac{3\pi}{4} < \frac{\theta}{2} < \pi$ Quad II

$\tan\frac{\theta}{2} = -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = -\sqrt{\frac{1-4/5}{1+4/5}\left(\frac{5}{5}\right)}$

$= -\sqrt{\frac{5-4}{5+4}} = -\sqrt{\frac{1}{9}} = -\frac{1}{3}$

Section 7.3: Verifying Identities

Let me verify the identity

$$\csc(x) - \sin(x) = \cos(x)\cot(x)$$

We'll work with the left side applying known identities to show it's equal to the right side.

$$\csc x - \sin x = \frac{1}{\sin x} - \sin x \qquad \text{reciprocal ID}$$

$$= \frac{1}{\sin x} - \frac{\sin^2 x}{\sin x} \qquad \text{algebra}$$

$$= \frac{1 - \sin^2 x}{\sin x} \qquad ''$$

$$= \frac{\cos^2 x}{\sin x}$$ Pythagorean ID

$$= \cos x \, \frac{\cos x}{\sin x}$$ algebra

$$= \cos x \, \cot x$$ quotient ID

Identity verified as expected.

Verifying Identities

Some things to note:

- Verifying an identity is **NOT** solving an equation.
- We do not ”do the same thing” to both sides.
- We do not assume the statement is true. We SHOW it!
- Pick one side, and apply identities to it. The goal is to transform it to the other side.
- Usually try to work with the *most complicated* side. (It’s usually easier to simplify a complicated expression than to complicate a simpler one!)
- Sometimes it helps to write everything in terms of sines and cosines—not always, but often.

Verify $\frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$

Starting with the left side

$$\frac{\sin x}{1-\cos x} = \left(\frac{\sin x}{1-\cos x}\right)\left(\frac{1+\cos x}{1+\cos x}\right)$$

we can get $1-\cos^2 x$ in the denominator

$$= \frac{\sin x\,(1+\cos x)}{1-\cos^2 x}$$

$$= \frac{\sin x\,(1+\cos x)}{\sin^2 x}$$

Pythagorean ID

$$= \frac{\sin x\,(1+\cos x)}{\sin x \sin x}$$

Cancel common factor

$$= \frac{1+\cos x}{\sin x}$$

as expected

Verify $\frac{\cos\theta\cot\theta}{1-\sin\theta} - 1 = \csc\theta$

$$\frac{\cos\theta\cot\theta}{1-\sin\theta} - 1 = \frac{\cos\theta\cot\theta}{1-\sin\theta}\left(\frac{1+\sin\theta}{1+\sin\theta}\right) - \frac{1-\sin^2\theta}{1-\sin^2\theta}$$

$$= \frac{\cos\theta\cot\theta(1+\sin\theta) - (1-\sin^2\theta)}{1-\sin^2\theta}$$

$$= \frac{\cos\theta\frac{\cos\theta}{\sin\theta}(1+\sin\theta) - 1 + \sin^2\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta\left(\frac{1}{\sin\theta} + 1\right) - \cos^2\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta\left(\frac{1}{\sin\theta}+1-1\right)}{\cos^2\theta}$$

$$= \frac{1}{\sin\theta}+1-1$$

$$= \frac{1}{\sin\theta}$$

$$= \csc\theta$$

as expected