November 26 MATH 1113 sec. 52 Fall 2018

Section 7.2: Double & Half Angle IDs

Use $\sin(u+v) = \sin u \cos v + \sin v \cos u$ to obtain a formula for

$\sin(2u)$ $\qquad\qquad 2u = u+u$

$= \sin(u+u)$

$= \sin u \cos u + \sin u \cos u$

$= 2\sin u \cos u$

Double Angle Formulas for the Cosine

Use $\cos(u+v) = \cos u \cos v - \sin u \sin v$ and $\cos^2 u + \sin^2 u = 1$ to find three formulas for

$\cos(2u)$ $= \cos(u+u) = \cos u \cos u - \sin u \sin u$

$= \cos^2 u - \sin^2 u$

Using $\sin^2 u = 1 - \cos^2 u$

$\cos(2u) = \cos^2 u - (1 - \cos^2 u)$

$= 2\cos^2 u - 1$

Using $\cos^2 u = 1 - \sin^2 u$

$$\cos(2u) = (1 - \sin^2 u) - \sin^2 u$$

$$= 1 - 2\sin^2 u$$

Question: Double Angle Formulas for the Tangent

From the sum formula $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$, it follows that

$\tan(2u) =$

(a) $\dfrac{2\tan u}{1 - 2\tan u}$

(b) $\dfrac{2\tan u}{1 - \tan^2 u}$

(c) $\dfrac{\tan^2 u}{1 - 2\tan u}$

(d) $\dfrac{\tan^2 u}{1 - \tan^2 u}$

Double Angle Formulas

$$
\begin{aligned}
\sin(2u) &= 2\sin u\cos u \\
\\
\cos(2u) &= \cos^2 u - \sin^2 u \\
&= 2\cos^2 u - 1 \\
&= 1 - 2\sin^2 u \\
\\
\tan(2u) &= \frac{2\tan u}{1-\tan^2 u}
\end{aligned}
$$

Example

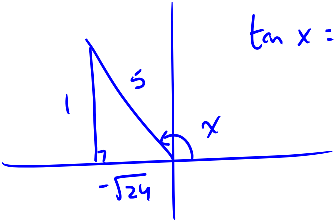
Suppose $\csc(x) = 5$ and $\cot(x) < 0$. Find the exact value of

(a) $\tan(2x)$

$\csc x > 0$ and $\cot x < 0$

x has terminal side in quad II in standard position

(b) $\sec(2x)$



$\tan x = \dfrac{-1}{\sqrt{24}}$

a) $\tan(2x) = \dfrac{2\tan x}{1-\tan^2 x}$

$= \dfrac{2\left(\frac{-1}{\sqrt{24}}\right)}{1-\left(\frac{-1}{\sqrt{24}}\right)^2}$

$$= \frac{\frac{-2}{\sqrt{24}}}{1 - \frac{1}{24}} \left(\frac{24}{24}\right) = \frac{-2\sqrt{24}}{24-1}$$

$$\tan(2x) = \frac{-2\sqrt{24}}{23}$$

$$\csc x = 5 \Rightarrow \sin x = \frac{1}{5}$$

b) $\sec(2x) = \dfrac{25}{23}$

$$\cos(2x) = 1 - 2\sin^2(x)$$

$$= 1 - 2\left(\frac{1}{5}\right)^2$$

$$= 1 - \frac{2}{25} = \frac{25-2}{25}$$

$$= \frac{23}{25}$$

Half Angle IDs

Use the fact that $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and that $\frac{\pi}{4} = 2\frac{\pi}{8}$ to find the exact value of

$\sin^2\left(\frac{\pi}{8}\right)$

$$\cos\left(\frac{\pi}{4}\right) = \cos\left(2\frac{\pi}{8}\right) = 1 - 2\sin^2\left(\frac{\pi}{8}\right)$$

$$2\sin^2\left(\frac{\pi}{8}\right) = 1 - \cos\left(\frac{\pi}{4}\right)$$

$$\sin^2\left(\frac{\pi}{8}\right) = \frac{1-\cos(\pi/4)}{2}$$

$$= \left(\frac{1-\frac{1}{\sqrt{2}}}{2}\right)\cdot\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{2}-1}{2\sqrt{2}}$$

$$= \frac{\sqrt{2}-1}{2\sqrt{2}}$$

Half Angle IDs

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan^2 x = \frac{1-\cos(2x)}{1+\cos(2x)} \qquad \tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{1+\cos(x)}}$$

For a given value of x, only one of the signs $+$ or $-$ will apply. To choose the correct sign, determine which quadrant the angle $\frac{x}{2}$ is in when in standard position.

Determine the exact value of

(a) $\cos(22.5^\circ)$

$$22.5^\circ = \frac{1}{2}(45^\circ)$$

$$0^\circ < 22.5^\circ < 90^\circ$$

so

$$\cos(22.5^\circ) > 0$$

$$= +\sqrt{\frac{1+\cos(2(22.5^\circ))}{2}}$$

$$= +\sqrt{\frac{1+\cos 45^\circ}{2}} = \sqrt{\left(\frac{1+\frac{1}{\sqrt{2}}}{2}\right)\cdot\frac{\sqrt{2}}{\sqrt{2}}}$$

$$= \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

Question

In standard position, the angle $\frac{13\pi}{12}$ would have its terminal side in

(a) Quadrant 1

(b) Quadrant 2

(c) Quadrant 3

(d) Quadrant 4

(e) $\frac{13\pi}{12}$ isn't a number, Dr. Ritter made it up

$$\frac{13\pi}{12} = \pi + \frac{\pi}{12}$$

Question

The angle $\frac{13\pi}{6}$ is coterminal with

(a) $\frac{\pi}{6}$

$$\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$$

(b) $\frac{5\pi}{6}$

(c) $\frac{7\pi}{6}$

(d) $\frac{11\pi}{6}$

(e) $\frac{13\pi}{6}$ isn't a number, Dr. Ritter made it up

Question

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$$

The exact value of $\sin\left(\frac{13\pi}{12}\right) = -\sqrt{\frac{1-\cos\left(\frac{13\pi}{6}\right)}{2}}$

$$= -\sqrt{\frac{1-\cos \pi/6}{2}}$$

(a) $\frac{1}{4}$

(b) $-\frac{\sqrt{3}}{4}$

(c) $-\frac{\sqrt{2-\sqrt{3}}}{2}$

(d) $\frac{\sqrt{2-\sqrt{3}}}{2}$

Find the exact value

(c) $\tan\left(\frac{\theta}{2}\right)$ given $\cos\theta = \frac{4}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

From $\frac{3\pi}{2} < \theta < 2\pi$

$$\frac{1}{2}\left(\frac{3\pi}{2}\right) < \frac{1}{2}\theta < \frac{1}{2}(2\pi)$$

$$\frac{3\pi}{4} < \frac{\theta}{2} < \pi$$

quad II

$\tan\frac{\theta}{2} < 0$

$$\tan\frac{\theta}{2} = -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = -\sqrt{\frac{1-4/5}{1+4/5}}\cdot\frac{5}{5}$$

$$= -\sqrt{\frac{5-4}{5+4}} = -\sqrt{\frac{1}{9}} = \frac{-1}{3}$$

Section 7.3: Verifying Identities

Let me verify the identity

$$\csc(x) - \sin(x) = \cos(x)\cot(x)$$

We'll use existing IDs to show that the left side equals the right side. Starting on the left side

$$\csc x - \sin x = \frac{1}{\sin x} - \sin x$$ reciprocal ID

$$= \frac{1}{\sin x} - \frac{\sin^2 x}{\sin x}$$ algebra

$$= \frac{1 - \sin^2 x}{\sin x}$$ algebra

$$= \frac{\cos^2 x}{\sin x}$$ Pythagorean ID

$$= \cos x \, \frac{\cos x}{\sin x}$$ algebra

$$= \cos x \cot x$$ quotient ID

Verified as expected.

Verifying Identities

Some things to note:

- Verifying an identity is **NOT** solving an equation.
- We do not ”do the same thing” to both sides.
- We do not assume the statement is true. We SHOW it!
- Pick one side, and apply identities to it. The goal is to transform it to the other side.
- Usually try to work with the *most complicated* side. (It’s usually easier to simplify a complicated expression than to complicate a simpler one!)
- Sometimes it helps to write everything in terms of sines and cosines—not always, but often.

Verify $\frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$

Starting on the left

$$\frac{\sin x}{1-\cos x} = \left(\frac{\sin x}{1-\cos x}\right)\left(\frac{1+\cos x}{1+\cos x}\right)$$

we'd like to use

$\sin^2 x + \cos^2 x = 1$

$$= \frac{\sin x\,(1+\cos x)}{1-\cos^2 x}$$

$$= \frac{\sin x\,(1+\cos x)}{\sin^2 x}$$

Pythagorean ID

$$= \frac{\sin x\,(1+\cos x)}{\sin x \; \sin x}$$

cancel common factor

$$= \frac{1+\cos x}{\sin x}$$

as expected.